Learning objectives:

1. Understand the definition of a definite integral.
2. Evaluating definite integral as limit of a sum.
3. Use areas to compute definite integrals.
4. Approximate definite integrals using the midpoint rule.
5. Learn the properties of definite integrals.

Definition of a definite integral

Let f be a function defined for $a \le x \le b$. Divide $[a, b]$ into n subintervals of equal width and choose sample points x_i^* on every subinterval. Then

$n(\Delta x) = b-a$

$\Delta x = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x\,,$$

$= \lim_{n\to\infty} R_n$

provided the above limit exists.

R_n same as in 4.1

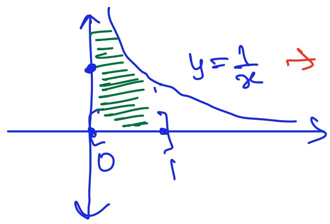
If the above limit exists we say f is integrable on $[a, b]$.

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*)}$$

$\int_a^b \underbrace{f(x)}_{\text{integrand}}\,dx$

$a, b \to$ limits of integration.

$\uparrow$ variable of integration



$\to$ not integrable on $(0, 1]$

Theorem

If f is continuous on $[a, b]$, or if f only a finite number of jump discontinuities, then f is integrable on $[a, b]$, that is, the definite integral $\int_a^b f(x)\,dx$ exists.

LHL = finite
RHL = finite
LHL $\ne$ RHL

Theorem

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x\,.$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$.

$[a, b] = [\underset{=a}{x_0}, \underset{=a+\Delta x}{x_1}] \cup [x_1, \underset{=a+2\Delta x}{x_2}] \cup \cdots \cup [\underset{=a+(n-1)\Delta x}{x_{n-1}}, x_n]$

$x_n = a + n(\Delta x) = b$

$x_i \to$ right endpoint of i^{th} subinterval

Properties of sums

$ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n)$

$$\sum_{i=1}^{n} c = nc\,, \qquad \sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i\,.$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\,.$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\,.$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\,, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\,, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}\,.$$

$1+2+3+\cdots+n$ $\quad = 1^2+2^2+3^2+\cdots+n^2$ $\quad = 1^3+2^3+3^3+\cdots+n^3$

Example 1. Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\,dx\,.$

Step 1 $\quad [0,3] \Rightarrow a=0,\ b=3 \Rightarrow \Delta x = \frac{3-0}{n} = \frac{3}{n}$

$x_i = a + i(\Delta x) = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$

Step 2 $\quad S_n = \sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right) = \sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$

$= \sum_{i=1}^{n}\left(\frac{3i}{n}\right)^3 - \sum_{i=1}^{n} 6\left(\frac{3i}{n}\right) = \sum_{i=1}^{n}\frac{27\,i^3}{n^3} - \sum_{i=1}^{n}\frac{18\,i}{n}$

$= \frac{27}{n^3}\sum_{i=1}^{n} i^3 - \frac{18}{n}\sum_{i=1}^{n} i = \frac{27}{n^3}\,\frac{\cancel{n^2}(n+1)^2}{4} - \frac{\overset{9}{\cancel{18}}}{\cancel{n}}\,\frac{\cancel{n}(n+1)}{\cancel{2}}$

$= \boxed{\frac{27}{4}\frac{(n+1)^2}{n} - 9(n+1)} = 9(n+1)\left[\frac{3(n+1)}{4n} - 1\right]$

Take limit

$= 9(n+1)\left[\frac{3(n+1) - 4n}{4n}\right] = \frac{9(n+1)(3n+3-4n)}{4n} = \frac{9(n+1)(-n+3)}{4n}$

Step 3 $\quad \int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \Delta x\, S_n = \lim_{n\to\infty} \frac{3}{n}\,\frac{9(n+1)(-n+3)}{4n}$

$= \lim_{n\to\infty} \frac{27(n+1)(-n+3)}{4n^2} = \lim_{n\to\infty} \frac{27}{4}\left(\frac{n+1}{n}\right)\left(\frac{-n+3}{n}\right)$

$$= \lim_{n\to\infty} \frac{27}{4}\left(1+\frac{1}{n}\right)\left(-1+\frac{3}{n}\right) = \frac{27}{4}(1+0)(-1+0) = -\frac{27}{4}$$

Example 2. Set up an expression for $\int_2^5 x^4\,dx$ as a limit of a sum.

Step 1 $[2,5] \Rightarrow a=2,\ b=5 \Rightarrow \Delta x = \frac{5-2}{n} = \frac{3}{n}$

$$x_i = a + i(\Delta x) = 2 + i\left(\frac{3}{n}\right) = 2 + \frac{3i}{n} = \frac{3i+2n}{n}$$

Step 2 $S_n = \sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} x_i^4 = \sum_{i=1}^{n}\left(\frac{3i+2n}{n}\right)^4$

$$= \sum_{i=1}^{n} \frac{(3i+2n)^4}{n^4}$$

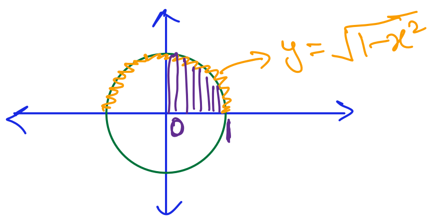
Step 3 $\int_2^5 x^4\,dx = \lim_{n\to\infty} (\Delta x) S_n = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{(3i+2n)^4}{n^4}$

$$= \lim_{n\to\infty} \frac{3}{n^5} \sum_{i=1}^{n} (3i+2n)^4$$

Example 3. Evaluate the following integrals by interpreting each in terms of areas.

1. $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$.

2. $\displaystyle\int_0^3 (x-1)\,dx$.



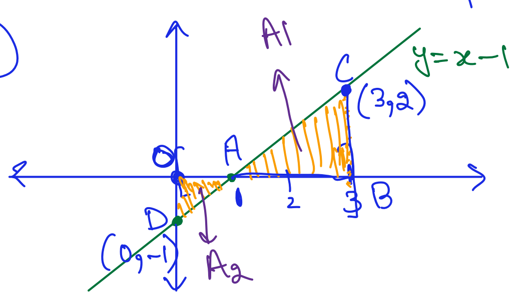
① $y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2+y^2 = 1$

$\int_0^1 \sqrt{1-x^2}\,dx$ = area under the curve $\underbrace{y=\sqrt{1-x^2}}_{\text{semi circle}}$ from 0 to 1

= area of a quarter of a circle

$= \frac{1}{4}$ (area of circle) $= \frac{1}{4}(\pi(1)^2) = \frac{\pi}{4}$

②



A_1 = area of ΔABC

$= \frac{1}{2}(AB)(BC) = \frac{1}{2}(2)(2) = 2$

A_2 = area of $\Delta OAD = \frac{1}{2}(OA)(OD)$

$= \frac{1}{2}(1)(1) = \frac{1}{2}$

$$\int_0^3 (x-1)\,dx = \int_0^1 (x-1)\,dx + \int_1^3 (x-1)\,dx$$

$= -A_2 + A_1 = -\frac{1}{2} + 2 = \frac{3}{2}$

$f(x_1) + f(x_2) + \cdots + f(x_n)$

$[a,b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n]$

$f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)$

The Midpoint rule

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x \approx \Delta x\big(f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\big)\,,$$

where $\Delta x = \frac{b-a}{n}$ and $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ is the midpoint of $[x_{i-1}, x_i]$.

Example 4. Use the midpoint rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\,dx$.

$[1,2] \Rightarrow a = 1,\ b = 2$, $\quad \Delta x = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5}$

$n = 5$

$x_i = a + i(\Delta x) = 1 + i\left(\frac{1}{5}\right) = 1 + \frac{i}{5}$

$x_0 = 1$, $\quad x_1 = 1 + \frac{1}{5} = \frac{6}{5}$, $\quad x_2 = 1 + \frac{2}{5} = \frac{7}{5}$, $\quad x_3 = 1 + \frac{3}{5} = \frac{8}{5}$,

$x_4 = 1 + \frac{4}{5} = \frac{9}{5}$, $\quad x_5 = 1 + \frac{5}{5} = \frac{10}{5}$

$\overline{x}_1 = \frac{x_0 + x_1}{2} = \frac{11}{10}$, $\quad \overline{x}_2 = \frac{x_1 + x_2}{2} = \frac{13}{10}$

$\overline{x}_3 = \frac{x_2 + x_3}{2} = \frac{15}{10}$, $\quad \overline{x}_4 = \frac{x_3 + x_4}{2} = \frac{17}{10}$, $\quad \overline{x}_5 = \frac{x_4 + x_5}{2} = \frac{19}{10}$

$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\left(f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4) + f(\overline{x}_5)\right)$

$\approx \frac{1}{5}\left[\frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19}\right] = \frac{10}{5}\left[\frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19}\right]$

$= 2\left[\frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19}\right] \approx 0.69$

Properties of definite integral

1. $\displaystyle\int_a^b c\,dx = c(b-a)$, where c is any constant.
2. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.
3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.
4. $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
5. $\displaystyle\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.
6. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, for $a<c<b$. → useful in case of jump discontinuities.
7. If $f(x)\geq 0$ for $a\leq x\leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$.
8. If $f(x)\geq g(x)$ for $a\leq x\leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
9. If $m\leq f(x)\leq M$ for $a\leq x\leq b$, then $\displaystyle m(b-a)\leq \int_a^b f(x)\,dx \leq M(b-a)$.

Example 5. Use the properties of integrals to evaluate $\int_0^1 (4+3x^2)\,dx$.

$$\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx = 4(1-0) + 3\int_0^1 x^2\,dx$$

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{1-0}{n}\sum_{i=1}^{n} x_i^2 , \qquad x_i = 0 + i\left(\frac{1-0}{n}\right) = \frac{i}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty}\frac{1}{n^{\cancel{3}\,2}}\,\frac{\cancel{n}(n+1)(2n+1)}{6} = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2} = \lim_{n\to\infty}\left(\frac{2}{6}+\frac{3}{6}\frac{1}{n}+\frac{1}{6n^2}\right) = \frac{1}{3}+0+0 = \frac{1}{3}$$

$$\int_0^1 (4+3x^2)\,dx = 4 + 3\left(\frac{1}{3}\right) = 5$$

Example 6. If $\int_0^5 f(x)\,dx = 7$ and $\int_0^3 f(x)\,dx = 2$, then find $\int_3^5 f(x)\,dx$.

$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx \qquad \text{[Property 6]}$$

$$7 = 2 + \int_3^5 f(x)\,dx$$

$$\Rightarrow \int_3^5 f(x)\,dx = 5$$

Example 7. Use property 9 to estimate $\int_1^4 \sqrt{x}\,dx$.

If $m \le \sqrt{x} \le M$ for $1 \le x \le 4$

then $m(4-1) \le \int_1^4 \sqrt{x}\,dx \le M(4-1)$

Want to find m, M.

$f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}} > 0$ for $x > 0$

$\Rightarrow f$ is increasing.

$m = f(1) = \sqrt{1} = 1$, $M = f(4) = \sqrt{4} = 2$

$$\Rightarrow 1(4-1) \le \int_1^4 \sqrt{x}\,dx \le 2(4-1)$$

$$3 \le \int_1^4 \sqrt{x}\,dx \le 6$$