Learning objectives:

1. Understand the fundamental theorem of calculus.
2. Apply the fundamental theorem to find derivatives of certain functions.
3. Apply the fundamental theorem to compute definite integrals.

Fundamental theorem of calculus I.

Let f be continuous on $[a, b]$ and define the function g by

$$g(x) = \int_a^x f(t)\,dt\,, \quad a \le x \le b\,.$$

Then

1. g is continuous on $[a, b]$.
2. g is differentiable on (a, b).
3. $g'(x) = f(x)$.

$g'(x) = \frac{d}{dx}(g(x))$

⊛ can replace x with any other variable (not t)

$\int_x^{x+h} f(t)\,dt \simeq f(x)h$ for very small values of h

$\int_x^{x+h} f(t)\,dt = g(x+h) - g(x)$

$\int_a^{x+h} f(t)\,dt = \underbrace{\int_a^x f(t)\,dt}_{g(x)} + \int_x^{x+h} f(t)\,dt$

$\Rightarrow \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x)$

Example 1. Find the derivative of the function $g(x) = \int_0^x \underbrace{\sqrt{1+t^2}}_{f(t)}\,dt$.

$g'(x) = f(x)$

$= \sqrt{1+x^2}$

$f \xrightarrow[\int_a^x f(t)\,dt]{\text{integrate}} g \xrightarrow[\text{w.r.t. } x]{\text{differentiate}} f$

$\Rightarrow$ Process of integration and process of differentiation are inverses to each other.

Example 2. Find the derivative of the function $g(x) = \int_1^{x^4} \sec t\,dt$.

$$g(x) = \int_1^{x^4} \sec t \, dt$$

Let $u = x^4 \Rightarrow g(u(x)) = \int_1^{u} \sec t \, dt$

$$\Rightarrow \frac{d}{dx}\left(g(u(x))\right) = \frac{d}{du}\left[\int_1^u \sec t \, dt\right] \frac{d}{dx}(u(x))$$

$$= \sec(u) \frac{d}{dx}(x^4)$$

$$g'(x) = 4x^3 \sec(x^4)$$

$$\boxed{\text{In general, for } g(x) = \int_a^{u(x)} f(t)\,dt, \quad g'(x) = f(u(x)) \cdot u'(x)}$$

Example 3. Find the derivative of the function $\int_x^0 \sqrt{1 + \sec t}\,dt$.

$$g(x) = \int_x^0 \sqrt{1+\sec t} \, dt$$

$$= -\int_0^x \sqrt{1+\sec t} \, dt$$

$$g'(x) = -\frac{d}{dx}\left[\int_0^x \sqrt{1+\sec t}\, dt\right] = -\sqrt{1+\sec x}$$

In general, for $g(x) = \int_{v(x)}^{u(x)} f(t)\,dt$, we have

$$g'(x) = f(u(x))\,u'(x) - f(v(x))\,v'(x)$$

$$\int_{v(x)}^{u(x)} f(t)\,dt = \int_{v(x)}^{a} f(t)\,dt + \int_a^{u(x)} f(t)\,dt = \int_a^{u(x)} f(t)\,dt - \int_a^{v(x)} f(t)\,dt$$

Example 4. Find the derivative of the function $\int_{\cos x}^{\sin x} \sqrt{1-s^2}\, ds,\ 0 \le x \le \pi/2$. Use it to compute the given integral in terms of x.

$$g(x) = \int_{\cos x}^{\sin x} \sqrt{1-s^2}\, ds$$

$$\Rightarrow g'(x) = \sqrt{1-\sin^2 x}\left(\frac{d}{dx}(\sin x)\right) - \sqrt{1-\cos^2 x}\left(\frac{d}{dx}(\cos x)\right)$$

$0 \le x \le \frac{\pi}{2}$ $\Downarrow$ $\sin x, \cos x$ are both +ve

$$= \cos x(\cos x) - \sin x(-\sin x)$$

$$= \cos^2 x + \sin^2 x = 1$$

$$\Rightarrow g'(x) = 1, \quad g\left(\frac{\pi}{2}\right) = \int_{\cos \pi/2}^{\sin \pi/2} \sqrt{1-s^2}\, ds$$

$$= \int_0^1 \sqrt{1-s^2}\, ds = \text{area of a quarter of unit circle}$$

$$= \frac{\pi}{4}$$

$$\rightarrow g'(x) = 1, \quad g(\pi/2) = \pi/4$$

$$g(x) = x + C \Rightarrow \frac{\pi}{4} = \frac{\pi}{2} + C$$

$$\Rightarrow C = -\frac{\pi}{4}$$

$$\Rightarrow \int_{\cos x}^{\sin x} \sqrt{1-s^2}\, ds = x - \frac{\pi}{4}$$

Fundamental theorem of calculus II.

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is any antiderivative of f.

$$g(x) = \int_a^x f(t)\,dt \quad \Rightarrow \quad g'(x) = f(x)$$

$\Rightarrow$ g is an antiderivative of $f(x)$

$$\Rightarrow g(x) = F(x) + C$$

$$\Rightarrow g(a) = F(a) + C \quad \text{and} \quad g(b) = F(b) + C$$

$$\Rightarrow g(b) - g(a) = F(b) - F(a) \Rightarrow \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = F(b) - F(a)$$

0

$$\underset{n \neq -1}{\mathrm{Ad}}(x^n) = \frac{x^{n+1}}{n+1}$$

Example 5. Evaluate the integral $\int_{-2}^{1} x^3\,dx$.

$$\int_{-2}^{1} x^3\,dx = F(1) - F(-2)$$

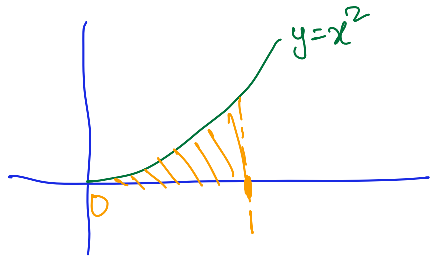
when F is an antiderivative of x^3

$$F = \frac{x^{3+1}}{3+1} = \frac{1}{4}x^4$$

$$\Rightarrow \int_{-2}^{1} x^3\,dx = \frac{1}{4}(1)^4 - \frac{1}{4}(-2)^4 = \frac{1}{4} - \frac{16}{4} = \frac{1}{4} - 4$$

$$= \frac{x^4}{4}\Big|_{-2}^{1} \qquad = -\frac{15}{4}$$

Example 6. Find the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$.

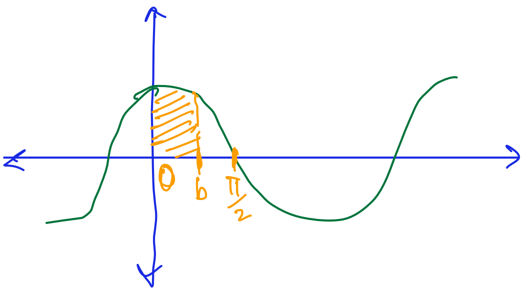


$$A = \int_0^1 x^2\,dx$$

$$= \frac{x^{2+1}}{2+1}\Big|_0^1 = \frac{x^3}{3}\Big|_0^1$$

$$= \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Example 7. Find the area under the cosine curve from $x = 0$ to $x = b$, where $0 \le b \le \pi/2$.



$$A = \int_0^b \cos x \, dx$$

$$= \sin x \Big|_0^b$$

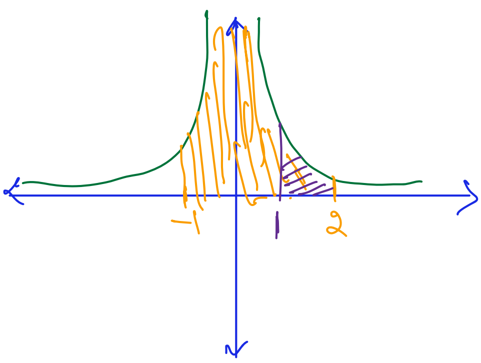
$$= \sin b - \sin 0$$

$$= \sin b$$

For $b = \frac{\pi}{2}$, $A = \sin \frac{\pi}{2} = 1$

Example 8. What is wrong with the following calculation?

$$\int_{-1}^{2} \frac{1}{x^2}\, dx = \frac{x^{-1}}{-1}\Big|_{-1}^{2} = -\frac{1}{2} - \frac{-1}{-1} = -\frac{3}{2}.$$



$f(x) = \frac{1}{x^2}$ is not continuous

$\Rightarrow$ Can not apply the fundamental theorem.

$$\int_1^2 \frac{1}{x^2}\, dx = \frac{x^{-1}}{-1}\Big|_1^2 = \frac{-1}{2} - \frac{1}{-1} = \frac{-1}{2} + 1 = \frac{1}{2}$$