Learning objectives:

1. The definition of indefinite integral
2. ~~Apply the fundamental theorem to find derivatives of certain functions~~.
3. Apply the fundamental theorem to compute definite integrals.
4. Net change theorem : integral of rate of change = Net change

$\int f'(x)\,dx = f(x) + c$

Indefinite integral

Ad(f) "

$$F(x) = \underbrace{\int f(x)\,dx}_{\text{antiderivative of } f} \quad \text{means} \quad F'(x) = f(x)$$

$\int F'(x)\,dx = F(x) + c$

Therefore, we have the following

$$\int cf(x)\,dx = c\int f(x)\,dx\,, \quad \int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx\,,$$

$$\int k\,dx = kx + c\,, \quad \int x^n\,dx = \frac{x^{n+1}}{n+1} + c\ (n \neq -1)\,,$$

$$\int \sin x\,dx = -\cos x + c\,, \quad \int \cos x\,dx = \sin x + c\,,$$

$$\int \sec^2 x\,dx = \tan x + c\,, \quad \int \csc^2 x\,dx = -\cot x + c\,,$$

$$\int \sec x \tan x\,dx = \sec x + c\,, \quad \int \csc x \cot x\,dx = -\csc x + c\,.$$

Example 1. Evaluate the indefinite integral $\int (10x^4 - 2\sec^2 x)\,dx$.

$$\int (10x^4 - 2\sec^2 x)\,dx = \int 10x^4\,dx - \int 2\sec^2 x\,dx$$

$$= 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\,\frac{x^{4+1}}{4+1} - 2\tan x + c$$

$$= \cancel{10}^{\,2}\,\frac{x^5}{\cancel{5}} - 2\tan x + c = 2x^5 - 2\tan x + c$$

Example 2. Evaluate $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\,d\theta$$

$$= \int \cot\theta\,\csc\theta\,d\theta$$

$$= -\csc\theta + C$$

Exercise ① $\displaystyle\int \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}\,d\theta = \int \tan\theta\cdot\sec\theta\,d\theta$

$$= \sec\theta + C$$

② $\displaystyle\int \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C$

③ $\displaystyle\int \frac{1}{\sin^2\theta}\,d\theta = \int \csc^2\theta\,d\theta$

$$= -\cot\theta + C$$

Example 3. Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

$$\int_0^3 (x^3 - 6x)\,dx = \int_0^3 x^3\,dx - \int_0^3 6x\,dx$$

$$= \int_0^3 x^3\,dx - 6\int_0^3 x\,dx$$

$$= \frac{x^4}{4}\Big|_0^3 - 6\,\frac{x^2}{2}\Big|_0^3$$

$$= \left(\frac{3^4}{4} - \frac{0^4}{4}\right) - 6\left(\frac{3^2}{2} - \frac{0^2}{2}\right)$$

$$= \frac{81}{4} - 6\cdot\frac{9}{2} = \frac{81}{4} - 27 = 27\left(\frac{3}{4} - 1\right)$$

$$= 27\left(\frac{-1}{4}\right) = -\frac{27}{4}$$

Example 4. Evaluate $\int_0^{12} (x - 12 \sin x)\, dx$.

$$\int_0^{12} (x - 12\sin x)\, dx$$

$$= \int_0^{12} x \, dx - \int_0^{12} 12 \sin x \, dx$$

$$= \frac{x^2}{2}\Big|_0^{12} - 12 \int_0^{12} \sin x \, dx$$

$$= \frac{12^2}{2} - 12 \left(-\cos x \Big|_0^{12} \right) = 72 - 12\left(-\cos 12 - (-\cos 0)\right)$$

$$= 72 - 12\left(-\cos 12 + 1\right)$$

$$= 72 + 12\cos 12 - 12$$

$$= 60 + 12\cos 12 = 12(5 + \cos 12)$$

Example 5. Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$.

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt = \int_1^9 \left(\frac{2\cancel{t^2}}{\cancel{t^2}} + \frac{\cancel{t^2}\sqrt{t}}{\cancel{t^2}} - \frac{1}{t^2} \right) dt$$

$$= \int_1^9 \left(2 + \sqrt{t} - t^{-2} \right) dt$$

$$= \int_1^9 2\, dt + \int_1^9 \sqrt{t}\, dt - \int_1^9 t^{-2}\, dt$$

$$= 2t\Big|_1^9 + \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_1^9 - \frac{t^{-1}}{-1}\Big|_1^9$$

$$= 2(9) - 2(1) + \frac{2}{3} 9^{3/2} - \frac{2}{3} 1^{3/2} - \left[\frac{9^{-1}}{-1} - \frac{1^{-1}}{-1} \right]$$

$$= 18 - 2 + \frac{2}{3} 3^{2 \cdot 3/2} - \frac{2}{3} - \left[\frac{-1}{9} + 1 \right] = 16 + 18 - \frac{2}{3} - \frac{8}{9}$$

$$\frac{292}{9}$$

$$\parallel$$

$$34 - \frac{14}{9}$$

$$\parallel$$

$$34 - \frac{6}{9} - \frac{8}{9}$$

$$//$$

The net change theorem

displacement from $t=a$, to $t=b$ $= \int_a^b v(t)\,dt$

The integral of a rate of change is the net change, that is,

$$\underbrace{\int_a^b F'(x)\,dx}_{\int \text{rate of change of } F} = \underbrace{F(b) - F(a)}_{\text{net change in } F}.$$

Example 6. A particle moves along a line with velocity at time t, $v(t) = t^2 - t - 6$ (measured in meters per second).

1. Find the displacement of the particle during the time period $1 \le t \le 4$.
2. Find the distance traveled during this time period.

① displacement from ($t=1$ to $t=4$) $= \int_1^4 v(t)\,dt$

$$= \int_1^4 (t^2 - t - 6)\,dt$$

$$= \int_1^4 t^2\,dt - \int_1^4 t\,dt - \int_1^4 6\,dt = \frac{t^3}{3}\Big|_1^4 - \frac{t^2}{2}\Big|_1^4 - 6t\Big|_1^4$$

$$= \underbrace{\frac{4^3}{3} - \frac{1^3}{3}} - \underbrace{\left(\frac{4^2}{2} - \frac{1^2}{2}\right)} - \underbrace{6(4-1)}$$

$$= \frac{63}{3} - \frac{15}{2} - 18 = 21 - \frac{15}{2} - 18 = 3 - \frac{15}{2} = \frac{-9}{2}$$

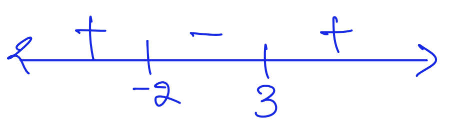
② distance is rate of change of speed

and speed is the absolute value of velocity.

$\Rightarrow$ distance in $1 \le t \le 4$ $= \int_1^4 |v(t)|\,dt$

$$|v(t)| = |t^2 - t - 6| = \begin{cases} -(t^2 - t - 6) & 1 \le t \le 3 \\ t^2 - t - 6 & 3 \le t \le 4 \end{cases}$$

$$t^2 - t - 6 = (t-3)(t+2)$$



$$\int_1^4 |t^2-t-6|\,dt = \int_1^3 -(t^2-t-6)\,dt + \int_3^4 (t^2-t-6)\,dt$$

$$= \int_1^3 (6+t-t^2)\,dt + \int_3^4 (t^2-t-6)\,dt$$

$$= 6t\Big|_1^3 + \frac{t^2}{2}\Big|_1^3 - \frac{t^3}{3}\Big|_1^3 + \frac{t^3}{3}\Big|_3^4 - \frac{t^2}{2}\Big|_3^4 - 6t\Big|_3^4$$

$$= 12 + 4 - \frac{26}{3} + \frac{37}{3} - \frac{7}{2} - 6$$

$$= 10 + \frac{11}{3} - \frac{7}{2} = \frac{60+22-21}{6} = \frac{61}{6} \text{ meters.}$$