Learning objectives:

1. Learn the chain rule analog for integration: called the substitution rule.
2. Apply the substitution rule to evaluate integrals.

If $F' = f$ then by chain rule $[F(g(x))]' = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$.

Letting $u = g(x)$ we get that the antiderivative of $f(g(x))\, g'(x)$ is given by $F(u)$, which is the antiderivative of $f(u)$.

The substitution rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, \underbrace{g'(x)\, dx} = \int f(u)\, \underbrace{du}.$$

Substitute $u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow du = \underbrace{g'(x)\, dx}$

Example 1. Evaluate the integral $\int 2x\sqrt{x^2+1}\, dx$.

Substitute $u = x^2+1$

$\Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\, dx$

$I = \int 2x\sqrt{x^2+1}\, dx = \int \sqrt{\underbrace{x^2+1}_{u}}\ \underbrace{2x\, dx}_{du}$

$= \int \sqrt{u}\, du = \int u^{1/2}\, du$

$= \frac{2}{3} u^{3/2} + C$

$\left(\int u^{1/2}\, du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C\right)$

Finally.

$I = \frac{2}{3}(x^2+1)^{3/2} + C$

Example 2. Evaluate $\int x^{3} \cos(x^4 + 2)\, dx$. $= \int x^3 \cos(x^4+2)\, dx$

Step 1

$u = x^4 + 2$

Step 2

$$\frac{du}{dx} = 4x^3 \Rightarrow du = 4x^3\, dx \Rightarrow \frac{1}{4}\, du = x^3\, dx$$

$$\Rightarrow I = \int x^3 \cos(x^4+2)\, dx = \int \cos(\underbrace{x^4+2}_{u})\, \underbrace{x^3\, dx}_{\frac{1}{4} du}$$

Step 3

$$= \int \cos u \cdot \frac{1}{4}\, du = \int \frac{1}{4} \cos u\, du = \frac{1}{4} \int \cos u\, du$$

$$= \frac{1}{4} \sin u + C$$

Step 4

$$\Rightarrow I = \frac{1}{4} \sin(x^4+2) + C$$

Example 3. Evaluate $\int \sqrt{2x + 1}\, dx$.

Step 1 $\quad u = 2x + 1$

Step 2 $\quad \frac{du}{dx} = 2 \Rightarrow du = 2\, dx \Rightarrow \frac{1}{2}\, du = dx$

$$\Rightarrow I = \int \sqrt{\underbrace{2x+1}_{u}}\, \underbrace{dx}_{\frac{1}{2} du} = \int \sqrt{u} \cdot \frac{1}{2} \cdot du = \int \frac{1}{2} \sqrt{u}\, du$$

Step 3

$$\Rightarrow I = \frac{1}{2} \int \sqrt{u}\, du = \frac{1}{2} \int u^{1/2}\, du$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2} + C = \frac{1}{3} u^{3/2} + C$$

Step 4

$$\Rightarrow I = \frac{1}{3} (2x+1)^{3/2} + C$$

Example 4. Evaluate $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Step1 $u = 1-4x^2$

Step2 $\frac{du}{dx} = -8x \Rightarrow du = -8x\,dx \Rightarrow \underbrace{\frac{-1}{8}\,du = x\,dx}$

$$\Rightarrow I = \int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{\underbrace{1-4x^2}_{u}}}\,\underbrace{x\,dx}_{\frac{-1}{8}du}$$

Step3

$$= \int \frac{1}{\sqrt{u}} \cdot \frac{-1}{8} \cdot du = \frac{-1}{8}\int u^{-\frac{1}{2}}\,du = \frac{-1}{8}\,\frac{u^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + C$$

$$= \frac{-1}{8} \cdot 2 \cdot u^{\frac{1}{2}} + C = \frac{-1}{4}\sqrt{u} + C$$

Step4 $\Rightarrow I = \frac{-1}{4}\sqrt{1-4x^2} + C$

Example 5. Evaluate $\int \cos 5x\,dx$.

Step1 $u = 5x$

Step2 $\frac{du}{dx} = 5 \Rightarrow du = 5\,dx \Rightarrow \frac{1}{5}\,du = dx$

Step3 $\Rightarrow I = \int \cos \underbrace{5x}_{u}\,\underbrace{dx}_{\frac{1}{5}du} = \int \cos u \cdot \frac{1}{5} \cdot du$

$$= \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C$$

Step4 $\Rightarrow I = \frac{1}{5}\sin(5x) + C$

In general,

$$\boxed{\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du}$$

Example 6. Evaluate $\int \sqrt{1+x^2}\, x^5\, dx$.

Step 1 $\quad u = 1+x^2$

Step 2 $\quad \dfrac{du}{dx} = 2x \Rightarrow du = 2x\, dx \Rightarrow \dfrac{1}{2}du = \underbrace{x\, dx}$

$$I = \int \sqrt{\underbrace{1+x^2}_{u}}\; x^4 \cdot \underbrace{x\, dx}_{\frac{1}{2}du} = \int \sqrt{u}\; \underbrace{x^4} \cdot \frac{1}{2}du$$

Express in terms of u

$$u = 1+x^2 \Rightarrow x^2 = u-1 \Rightarrow x^4 = (x^2)^2 = (u-1)^2$$

$$\Rightarrow I = \int \frac{1}{2}\sqrt{u}\,(u-1)^2\, du = \frac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\, du$$

$$= \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du = \frac{1}{2}\int u^{5/2}\, du + \frac{1}{2}\int -2u^{3/2}\, du + \frac{1}{2}\int u^{1/2}\, du$$

$$= \frac{1}{2}\frac{u^{7/2}}{7/2} - \frac{u^{5/2}}{5/2} + \frac{1}{2}\frac{u^{3/2}}{3/2} + C = \frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C$$

Step 4

$$\Rightarrow I = \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$$

Example 7. Evaluate $\int \sqrt{\cot x}\, \csc^2 x\, dx$.

Substitute $u = \cot x$

$$\Rightarrow \frac{du}{dx} = -\csc^2 x \Rightarrow du = -\csc^2 x\, dx \Rightarrow -du = \csc^2 x\, dx$$

$$I = \int \sqrt{\underbrace{\cot x}_{u}}\; \underbrace{\csc^2 x\, dx}_{-du} = \int \sqrt{u}\,(-du) = -\int u^{1/2}\, du$$

$$= -\frac{2}{3}u^{3/2} + C$$

$$= -\frac{2}{3}(\cot x)^{3/2} + C$$

The substitution rule for definite integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du\,.$$

$u = g(x) \Rightarrow$ a changes to $g(a)$ and b changes to $g(b)$

Example 8. Evaluate $\int_0^1 \cos(\pi t/2)\, dt$.

Step 1 $u = \frac{\pi t}{2}$ Step 2 $\frac{du}{dt} = \frac{\pi}{2} \Rightarrow du = \frac{\pi}{2}\, dt$

$\Rightarrow \frac{2}{\pi}\, du = dt$

Step 3 $\Rightarrow I = \int_0^1 \cos\Big(\underbrace{\frac{\pi t}{2}}_{u}\Big)\, \underbrace{dt}_{\frac{2}{\pi}du} = \int_{\frac{\pi}{2}\cdot 0}^{\frac{\pi}{2}\cdot 1} \cos u \cdot \frac{2}{\pi}\, du$

Step 4 $= \int_0^{\pi/2} \frac{2}{\pi} \cos u\, du = \frac{2}{\pi}\int_0^{\pi/2} \cos u\, du = \frac{2}{\pi} \sin u \Big|_0^{\frac{\pi}{2}}$

$= \frac{2}{\pi}\left[\sin\frac{\pi}{2} - \sin 0\right] = \frac{2}{\pi}[1 - 0] = \frac{2}{\pi}$

Alternatively

$\int_0^1 \cos\left(\frac{\pi t}{2}\right) dt = \frac{2}{\pi} \sin\left(\frac{\pi t}{2}\right)\Big|_0^1 = \frac{2}{\pi}\sin\frac{\pi}{2} - \frac{2}{\pi}\sin 0 = \frac{2}{\pi}$

Example 9. Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

Substitute $u = 3 - 5x \Rightarrow \frac{du}{dx} = -5 \Rightarrow du = -5\, dx$

$\Rightarrow -\frac{1}{5}\, du = dx$

$\Rightarrow I = \int_1^2 \frac{dx}{(3-5x)^2} = \int_1^2 \frac{1}{(\underbrace{3-5x}_{u})^2}\, \underbrace{dx}_{-\frac{1}{5}du}$

$= \int_{3-5(1)}^{3-5(2)} \frac{1}{u^2} \cdot \frac{-1}{5}\, du = -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\, du$

$$\Rightarrow I = \frac{-1}{5}\int_{-2}^{-7} u^{-2}\,du = \frac{-1}{5}\left[\frac{u^{-2+1}}{-2+1}\Bigg|_{-2}^{-7}\right]$$

$$= \frac{-1}{5}\left[\frac{u^{-1}}{-1}\Bigg|_{-2}^{-7}\right] = \frac{-1}{5}\left[\frac{-1}{-7} - \frac{-1}{-2}\right] = \frac{-1}{5}\left[\frac{1}{7} - \frac{1}{2}\right]$$

$$= \frac{-1}{5}\left(\frac{2-7}{14}\right) = \frac{-1}{\cancel{5}}\cdot\frac{-\cancel{5}}{14} = \frac{1}{14}$$

Symmetry

Let f be continuous on $[-a, a]$.

1. If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
2. If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

① $f(-x) = f(x)$

$$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{\substack{u=-x \;\Rightarrow\; x=-u \\ \Rightarrow du = -dx}} + \int_{0}^{a} f(x)\,dx$$

$$\int_{a}^{0} f(u)\,du = -\int_{0}^{a} f(-u)\,du$$

$$\int_{-a}^{0} f(x)\,dx = \int_{-(-a)}^{0} f(-u)\,(-du) = -\int_{a}^{0} f(-u)\,du = \int_{0}^{a} f(-u)\,du$$

f even

$$= \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

② $f(-x) = -f(x)$ following the same steps as in ①

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} -f(u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Example 10. Evaluate the following integrals.

1. $\int_{-2}^{2} (x^6 + 1)\,dx$.
2. $\displaystyle\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx$.

① $\int_{-2}^{2} \underbrace{(x^6+1)}\,dx$

$f(x) = x^6+1 \Rightarrow f(-x) = (-x)^6+1 = x^6+1 = f(x)$

$\Rightarrow f$ is an even function.

$$\int_{-2}^{2} (x^6+1)\,dx = 2\int_{0}^{2} (x^6+1)\,dx$$

$$= 2\left[\int_0^2 x^6\,dx + \int_0^2 1\,dx\right]$$

$$= 2\left[\frac{x^7}{7}\Big|_0^2 + x\Big|_0^2\right] = 2\left[\frac{2^7}{7} + 2\right]$$

$$= \frac{2^8}{7} + 4 = \frac{256}{7} + 4 = 36.57 + 4 = 40.57$$

② $\displaystyle\int_{-1}^{1} \underbrace{\frac{\tan x}{1+x^2+x^4}}\,dx$

$$f(x) = \frac{\tan x}{1+x^2+x^4} \Rightarrow f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4}$$

$$= \frac{-\tan x}{1+x^2+x^4} = -f(x)$$

$\Rightarrow f(-x) = -f(x) \Rightarrow f$ is odd

$$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx = 0$$