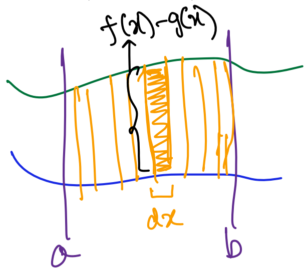
Learning objectives:

1. Find areas of regions bounded between two or more curves.
2. We either divide a region in vertical strips and integrate with respect to x, or we divide a region in horizontal strips and integrate with respect to y.

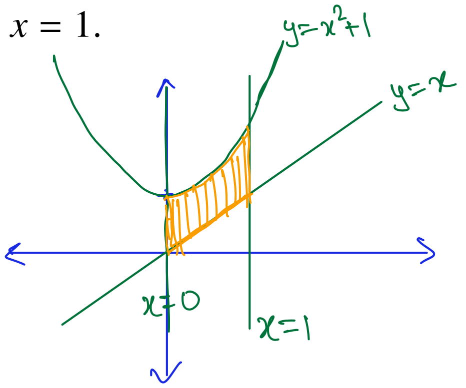
Area using vertical strips

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$, for $a \leq x \leq b$, is



$$A = \int_a^b \overbrace{[\underbrace{f(x)}_{\text{upper curve}} - \underbrace{g(x)}_{\text{lower curve}}]\, dx}^{\text{area of one very small vertical strip.}} .$$

Example 1. Find the area of the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$ and $x = 1$.



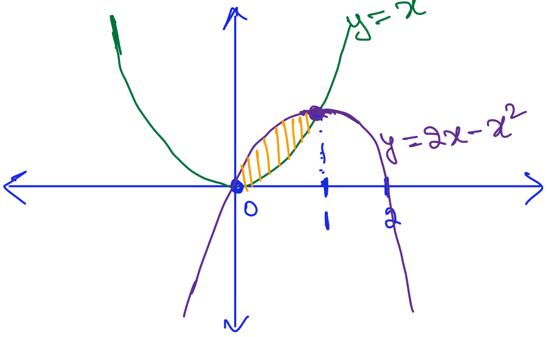
$$\Rightarrow A = \int_0^1 (\underbrace{x^2+1}_{\text{upper curve}} - \underbrace{x}_{\text{lower curve}})\, dx$$

$$\Rightarrow A = \int_0^1 x^2\, dx + \int_0^1 1\, dx - \int_0^1 x\, dx$$

$$= \frac{x^3}{3}\Big|_0^1 + x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1$$

$$= \frac{1}{3} + 1 - \frac{1}{2} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

Example 2. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.



For $y = ax^2 + bx + c$ the vertex lies at $x = \frac{-b}{2a}$

To find pts. of intersection solve $y = x^2$, $y = 2x - x^2$

$x^2 = 2x - x^2 \Rightarrow 2x^2 = 2x$

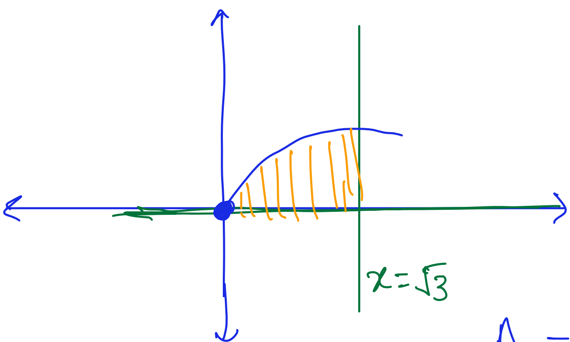
$\Rightarrow x = 0$ or $x = 1$

$$A = \int_0^1 \Big(\underbrace{2x - x^2}_{\text{upper curve}} - \underbrace{x^2}_{\text{lower curve}}\Big)\,dx = \int_0^1 (2x - 2x^2)\,dx$$

$$= 2\int_0^1 (x - x^2)\,dx = 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right]$$

$$= 2\left[\frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1\right] = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}$$

Example 3. Find the area of the region enclosed by $y = x/\sqrt{x^2+1}$, $x = \sqrt{3}$ and the x-axis.



At $x = 0$, $y = 0$

For $x > 0$, $y > 0$

$\Rightarrow y = \frac{x}{\sqrt{x^2+1}}$ is the upper curve

and $y = 0$ is the lower curve

$$A = \int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx$$

Let $u = x^2 + 1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$$A = \int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx = \int_0^{\sqrt{3}} \frac{1}{\underbrace{\sqrt{x^2+1}}_{u}} \cdot \underbrace{x\,dx}_{\frac{1}{2}du} = \int_{1+0^2}^{1+\sqrt{3}^2} \frac{1}{\sqrt{u}}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_1^4 u^{-1/2}\,du = \cancel{\frac{1}{2}}\,\frac{u^{-\frac{1}{2}+1}}{\cancel{\left(\frac{-1}{2}+1\right)}}\Bigg|_1^4 = \sqrt{u}\Big|_1^4 = \sqrt{4} - \sqrt{1} = 2 - 1 = 1$$

To find the area between the curves $y = f(x)$ and $y = g(x)$, when $f(x) \geq g(x)$ for some values of x while $g(x) \geq f(x)$ for some other values of x, we split the given region into several regions.

In general, the area between the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$, $(a < b)$, is given by

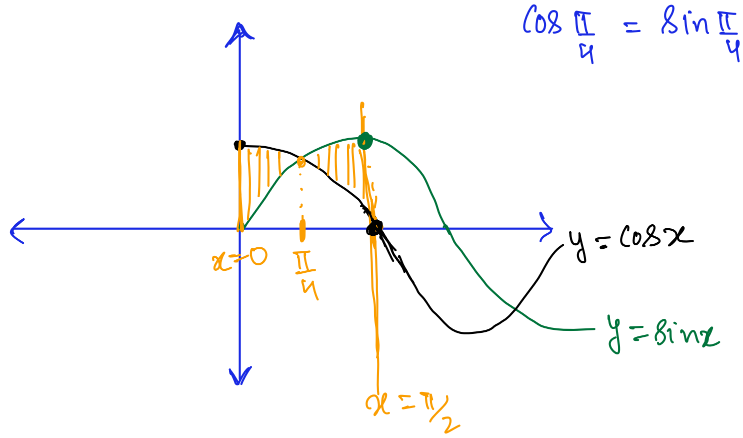
$$A = \int_a^b |f(x) - g(x)|\, dx\,.$$

Here we keep in mind that

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x)\,, \\ g(x) - f(x) & \text{if } g(x) \geq f(x)\,. \end{cases}$$

Example 4. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.

$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$



$$A = \int_0^{\frac{\pi}{2}} |\sin x - \cos x|\, dx = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \overbrace{\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right)}^{\frac{2}{\sqrt{2}} = \sqrt{2}} - (\sin 0 + \cos 0) + \left(-\cos\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\right) - \overbrace{\left(-\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4}\right)}^{-\frac{2}{\sqrt{2}} = -\sqrt{2}}$$

$$= \sqrt{2} - 1 + (-0 - 1) - (-\sqrt{2}) = 2\sqrt{2} - 2 \approx 0.828$$

Area using horizontal strips.

Some regions are best treated by regarding x as a function of y.

If a region is bounded by the curves $x = f(y)$, $x = g(x)$, $y = c$ and $y = d$, $(c < d)$, then its area is given by

$$A = \int_c^d |f(y) - g(y)|\, dy\ .$$

$$|f(y)-g(y)| = \begin{cases} f(y) - g(y) & f(y) \geq g(y) \\ g(y) - f(y) & g(y) \geq f(y) \end{cases}$$

Example 5. Find the area enclosed by the line $y = x-1$ and the parabola $\underbrace{y^2 = 2x+6}$.

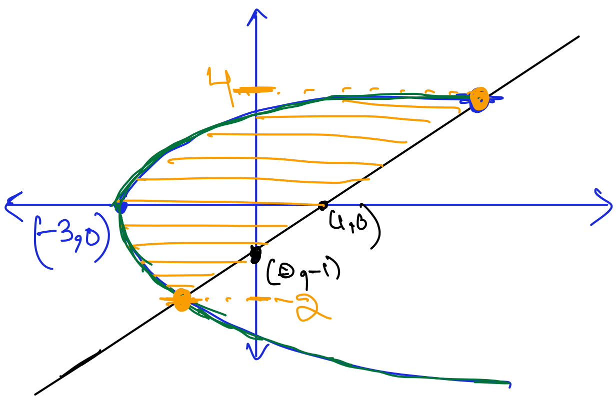
$y = \pm\sqrt{2x+6}$

Not a function (of x)

$\boxed{x = y+1} \Rightarrow f(y) = y+1$

$y^2 = 2x+6 \Rightarrow y^2 - 6 = 2x \Rightarrow \boxed{x = \frac{1}{2}y^2 - 3}$

$g(y) = \frac{1}{2}y^2 - 3$



Pts. of intersection

$x = y+1$

$x = \frac{1}{2}y^2 - 3$

$y+1 = \frac{1}{2}y^2 - 3$

$2y + 2 = y^2 - 6 \Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow y^2 - 4y + 2y - 8 = 0$

$\Rightarrow y(y-4) + 2(y-4) = 0$

$\Rightarrow (y-4)(y+2) = 0$

$\Rightarrow y = 4$ or $y = -2$

$$A = \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3\right)\right] dy$$

$$= \int_{-2}^{4} \left(y + 1 - \frac{1}{2}y^2 + 3\right) dy$$

$$= \int_{-2}^{4} \left(4 + y - \frac{1}{2}y^2\right) dy = 4y\Big|_{-2}^{4} + \frac{y^2}{2}\Big|_{-2}^{4} - \frac{1}{6}y^3\Big|_{-2}^{4}$$

$$= 4(6) + \frac{1}{2}(16-4) - \frac{1}{6}(64-(-8))$$

$$= 24 + 6 - \frac{1}{6}(72) = 30 - 12 = \boxed{18}$$