Learning objectives:

1. Find volumes of solids of revolution, obtained by revolving a region about a line called axis.
2. We divide the given solid into disks/washer by cutting it into infinite infinitesimally small cross-sections (region perpendicular to the axis of rotation).

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-section area of S in the plain P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \int_a^b A(x)\, dx \, .$$

We use the above formula when a solid is obtained by rotating a region about an axis which is parallel to the x-axis.

$$A(x) = \begin{cases} \pi\,(r(x))^2 & \text{disk method} \\ \pi\left[(r_{outer}(x))^2 - (r_{inner}(x))^2\right] & \text{washer method} \end{cases}$$

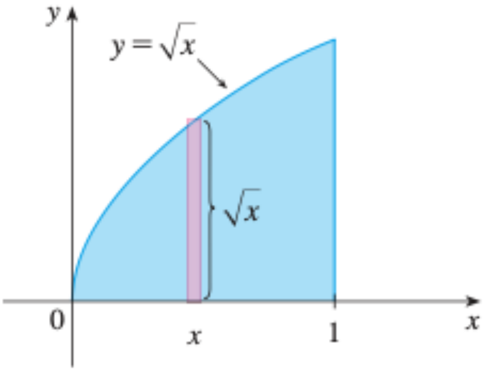
Let S be a solid that lies between $y = a$ and $y = b$. If the cross-section area of S in the plain P_y, through y and perpendicular to the y-axis, is $A(y)$, where A is a continuous function, then the volume of S is

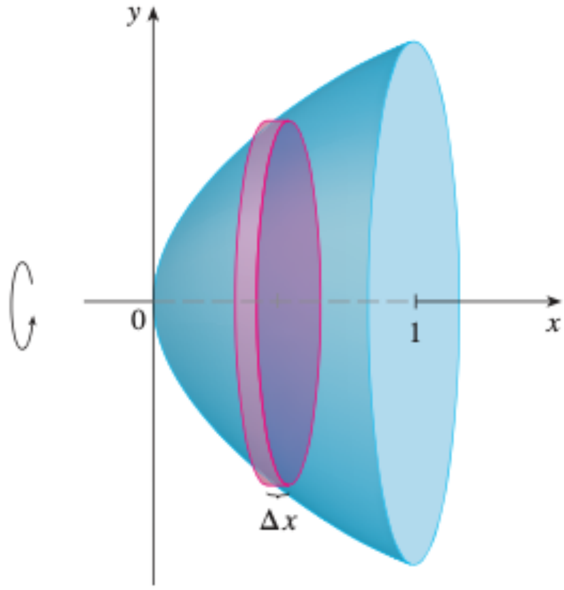
$$V = \int_a^b A(y)\, dy \, .$$

We use the above formula when a solid is obtained by rotating a region about an axis which is parallel to the y-axis.

$$A(y) = \begin{cases} \pi\,(r(y))^2 & \text{disk method} \\ \pi\left[(r_{outer}(y))^2 - (r_{inner}(y))^2\right] & \text{washer method} \end{cases}$$

Example 1. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$.





$$A(x) = \pi \left(r(x)\right)^2$$

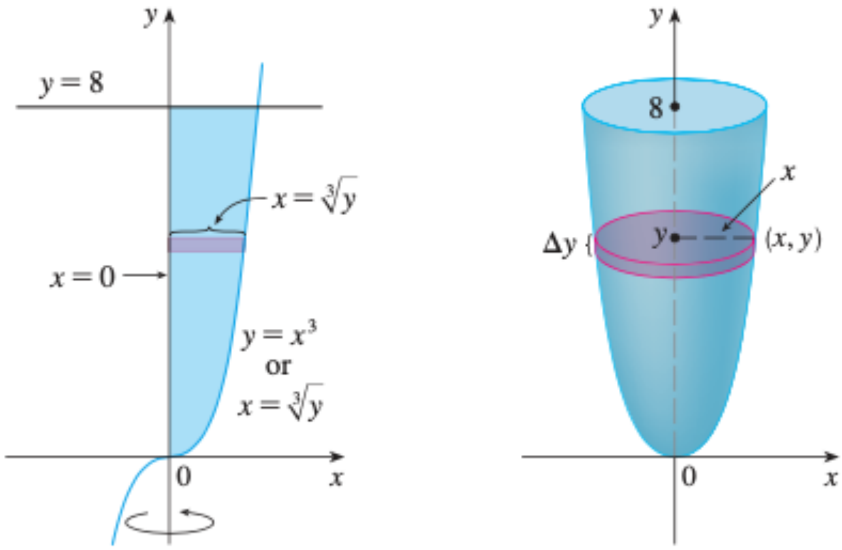
$$r(x) = \sqrt{x}$$

$$A(x) = \pi \left(\sqrt{x}\right)^2 = \pi x$$

$$V = \int_0^1 A(x)\, dx = \int_0^1 \pi x \, dx$$

$$= \pi \left.\frac{x^2}{2}\right|_0^1 = \frac{\pi}{2}$$

Example 2. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.



$$A(y) = \pi (r(y))^2 = \pi x^2$$

$$y = x^3 \Rightarrow x = \sqrt[3]{y} = r(y)$$

$$A(y) = \pi \left(\sqrt[3]{y}\right)^2 = \pi \, y^{2/3}$$

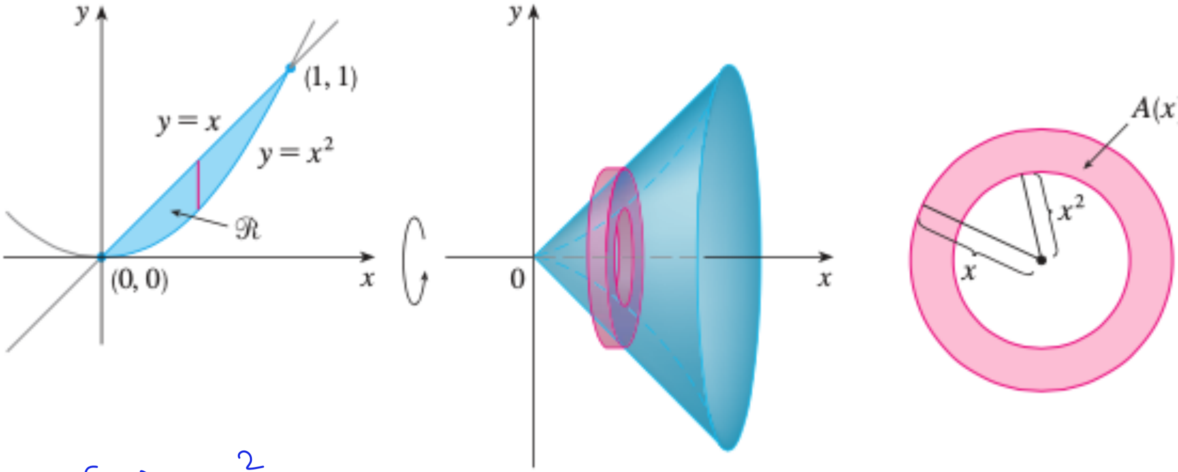
$$V = \int_0^8 \pi \, y^{2/3} \, dy$$

$$= \pi \int_0^8 y^{2/3} \, dy = \pi \left. \frac{y^{1+2/3}}{1+2/3} \right|_0^8$$

$$= \pi \left. \frac{y^{5/3}}{5/3} \right|_0^8 = \frac{3\pi}{5} \left[8^{5/3} - 0^{5/3} \right]$$

$$= \frac{3\pi}{5} \left[(2^3)^{5/3} \right] = \frac{3\pi}{5} 2^5 = \frac{96\pi}{5}$$

Example 3. The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.



$r_1(x) = x^2$

$r_2(x) = x$

$$A(x) = \pi\left(r_2^2 - r_1^2\right)$$
$$= \pi\left[(x)^2 - (x^2)^2\right]$$
$$= \pi\left(x^2 - x^4\right)$$

$$\Rightarrow V = \int_0^1 \pi\left(x^2 - x^4\right)dx$$

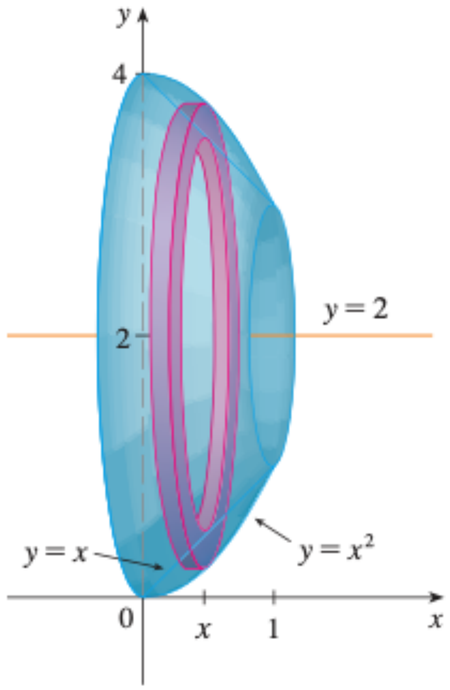
$f(x) = x$

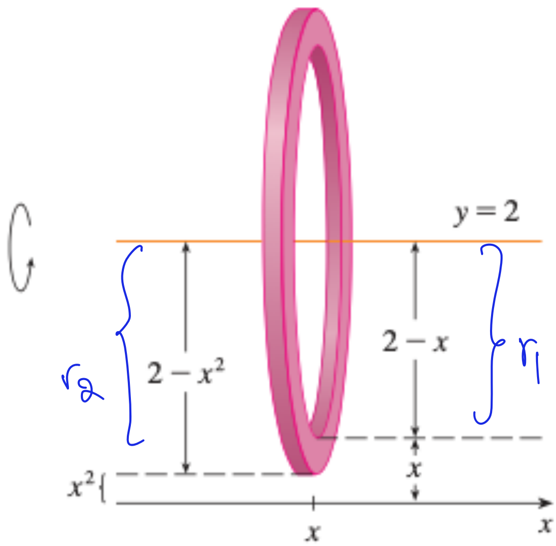
$g(x) = x^2$

$$V = \int_0^1 \pi\left[(f(x))^2 - (g(x))^2\right]dx$$

$$= \pi\int_0^1 \left(x^2 - x^4\right)dx$$
$$= \pi\left[\int_0^1 x^2\,dx - \int_0^1 x^4\,dx\right]$$
$$= \pi\left[\frac{x^3}{3}\Big|_0^1 - \frac{x^5}{5}\Big|_0^1\right]$$
$$= \pi\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{2\pi}{15}$$

Example 4. The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the $y = 2$ line. Find the volume of the resulting solid.





$$r_1 = 2 - x \quad , \quad r_2 = 2 - x^2$$

$$A(x) = \pi\left(r_2^2 - r_1^2\right)$$

$$= \pi\left[(2-x^2)^2 - (2-x)^2\right]$$

$$= \pi\left[4 - 4x^2 + x^4 - (4 - 4x + x^2)\right]$$

$$= \pi\left[\cancel{4} - 4x^2 + x^4 \cancel{-4} + 4x - x^2\right]$$

$$= \pi\left[4x - 5x^2 + x^4\right]$$

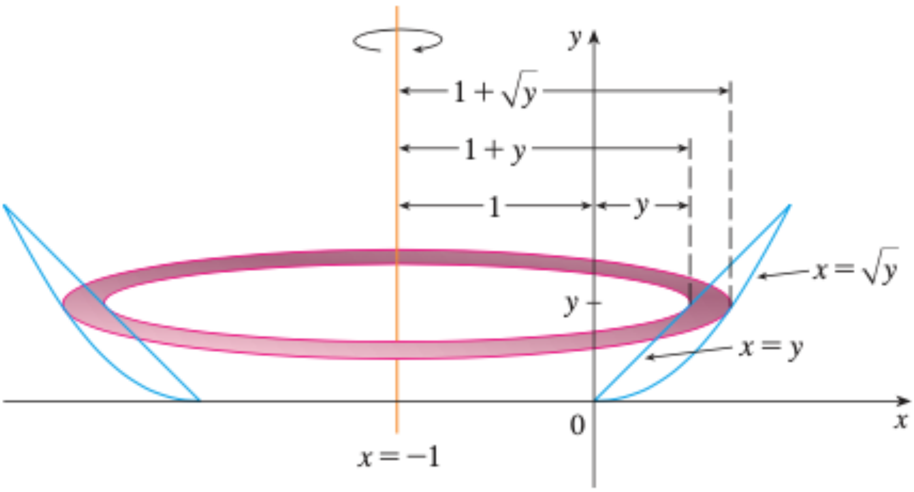
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left(4x - 5x^2 + x^4\right) dx$$

$$= \pi \int_0^1 \left(4x - 5x^2 + x^4\right) dx$$

$$= 4\pi \int_0^1 x\,dx - 5\pi \int_0^1 x^2\,dx + \pi \int_0^1 x^4\,dx = \frac{4\pi}{2} - \frac{5\pi}{3} + \frac{\pi}{5}$$

$$= 2\pi - \frac{5}{3}\pi + \frac{\pi}{5} = \frac{\pi}{3} + \frac{\pi}{5} = \frac{8\pi}{15}$$

Example 5. The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the $x = -1$ line. Find the volume of the resulting solid.



$$r_1 = 1+y \,, \qquad r_2 = 1+\sqrt{y}$$

$$A(y) = \pi\left(r_2^2 - r_1^2\right) = \pi\left[(1+\sqrt{y})^2 - (1+y)^2\right]$$

$$= \pi\left[1+2\sqrt{y}+y-(1+2y+y^2)\right]$$

$$= \pi\left[\cancel{1+}\,2\sqrt{y}+\underbrace{y}\,\cancel{-1}\,\underbrace{-2y}-y^2\right]$$

$$= \pi\left[2\sqrt{y}-y-y^2\right]$$

$$V = \int_0^1 \pi\left(2\sqrt{y}-y-y^2\right)dy$$

$$= 2\pi\int_0^1 \sqrt{y}\,dy - \pi\int_0^1 y\,dy - \pi\int_0^1 y^2\,dy$$

$$= 2\pi \left.\frac{y^{3/2}}{3/2}\right|_0^1 - \pi \left.\frac{y^2}{2}\right|_0^1 - \pi\left.\frac{y^3}{3}\right|_0^1$$

$$= 2\pi\left(\frac{2}{3}\right) - \frac{\pi}{2} - \frac{\pi}{3} = \frac{4\pi}{3} - \frac{5\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$$