Learning objectives:

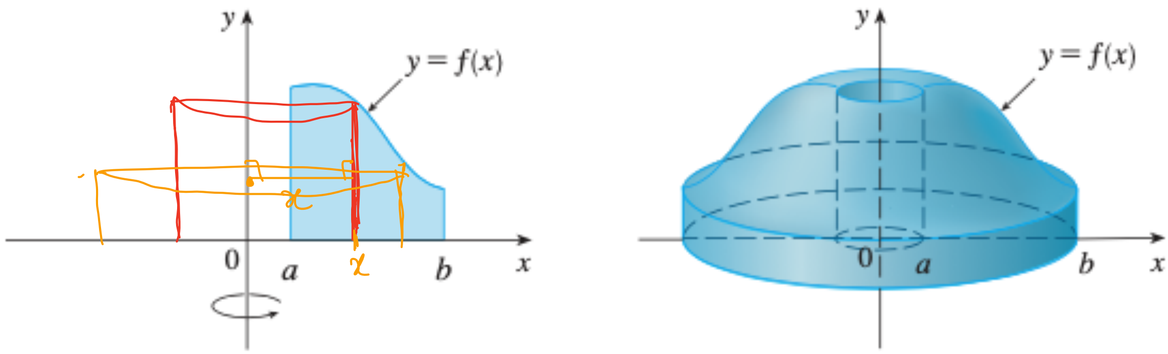
1. Find volumes of solids of revolution, obtained by revolving a region about a line called axis.
2. We divide the given solid into infinite cylinderical shells of infinitesimally small thickness.

The volume of a thin cylinderical shell of radius r and height h is given by

$$dV = 2\pi\, r\, h\, dr\ .$$

The volume of the solid shown in figure below, obtained by rotating the region on the left (region under $y = f(x)$ from a to b) about the y-axis, is

$$V = \int_a^b 2\pi\, x\, f(x)\, dx\ .$$

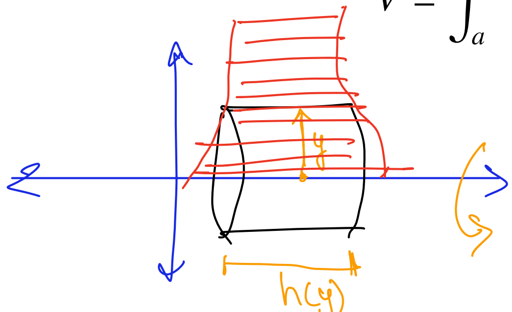


In general for a region bounded between $y = f(x)$ and $y = g(x)$ between $x = a$ to $x = b$, the volume of solid obtained by rotating it about the y-axis, is

$$V = \int_a^b 2\pi\, x\, |f(x) - g(x)|\, dx\ .$$

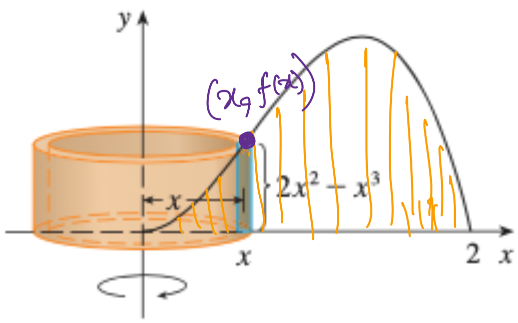
For a region bounded between $x = f(y)$ and $x = g(y)$ between $y = a$ to $y = b$, the volume of solid obtained by rotating it about the x-axis, is

$$V = \int_a^b 2\pi\, y\, |f(y) - g(y)|\, dy\ .$$

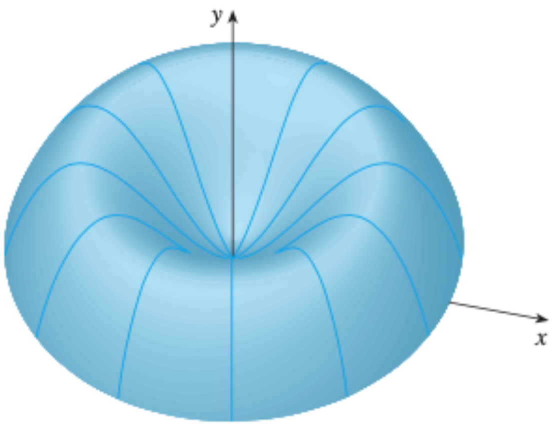


Example 1. Find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$, about the y-axis.

$$y = 2x^2 - x^2$$
$$= -x^2(x-2)$$



$$V = \int_0^2 2\pi x \;(2x^2 - x^3)\,dx$$



$$= 2\pi \int_0^2 (2x^3 - x^4)\,dx$$

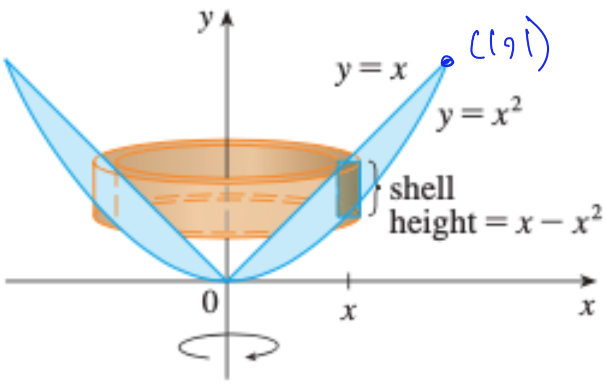
$$= 2\pi \left[2\frac{x^4}{4} - \frac{x^5}{5} \right]\Bigg|_0^2$$

$$= 2\pi \left[\frac{2}{4}(2)^4 - \frac{1}{5}(2^5) \right]$$

$$= 2\pi (2^5) \left[\frac{1}{4} - \frac{1}{5} \right]$$

$$= 2\pi (\overset{8}{\cancel{32}}) \frac{1}{\underset{5}{\cancel{20}}} = \boxed{\frac{16\pi}{5}}$$

Example 2. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.



$r = x$

$h = (x - x^2)$

$$V = \int_0^1 2\pi x (x - x^2)\, dx$$

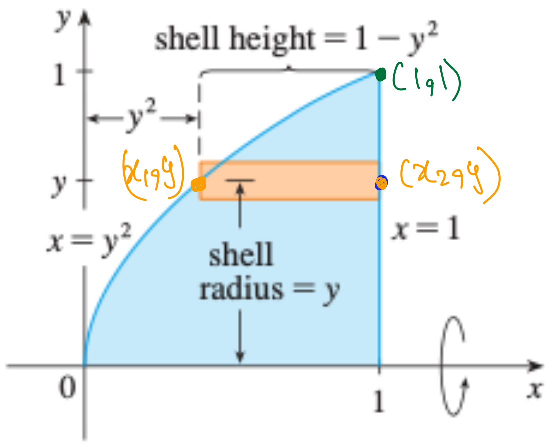
$$= 2\pi \int_0^1 x(x - x^2)\, dx$$

$$= 2\pi \int_0^1 (x^2 - x^3)\, dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]\Bigg|_0^1$$

$$= 2\pi \left[\frac{1}{3} - \frac{1}{4} \right] = 2\pi \left(\frac{1}{12} \right) = \frac{\pi}{6}$$

Example 3. The region R enclosed by the curves $y = \sqrt{x}$ and $y = 0$ is rotated about the x-axis. Find the volume of the resulting solid using cylinderical shell method.



$r = y$

$h = 1 - y^2 = x_2 - x_1$

$x_2 = 1$

$\sqrt{x_1} = y \Rightarrow x_1 = y^2$

$$dV = 2\pi y (1-y^2)\, dy$$

$$V = \int_0^1 2\pi y (1-y^2)\, dy$$

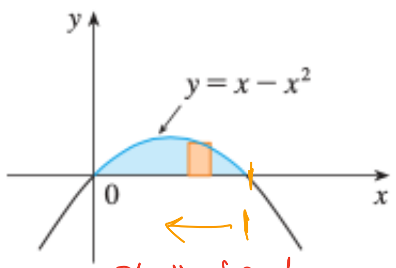
$$= 2\pi \int_0^1 y(1-y^2)\, dy$$

$$= 2\pi \left[\int_0^1 (y - y^3)\, dy \right]$$

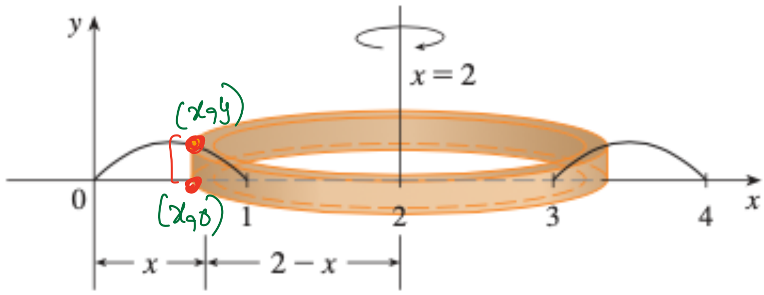
$$= 2\pi \left(\frac{y^2}{2} - \frac{y^4}{4} \right) \Big|_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$= 2\pi \left(\frac{1}{4} \right) = \boxed{\frac{\pi}{2}}$$

Example 4. The region R enclosed by the curves $y = x - x^2$ and $y = 0$ is rotated about the $x = 2$ line. Find the volume of the resulting solid.



x varies from $x=1$ to $x=0$



$$r = 2-x$$

$$h = y = x - x^2$$

$$dV = 2\pi(2-x)(x-x^2)\,dx$$

$$V = \int_{1}^{0} 2\pi(2-x)(x-x^2)(-dx)$$

0 ← farthest to axis.

1 ↑ nearest to axis

$$V = \int_0^1 2\pi(2-x)(x-x^2)\,dx$$

$$= 2\pi\int_0^1 \left[2x - 2x^2 - x^2 + x^3\right] dx$$

$$= 2\pi\int_0^1 \left(2x - 3x^2 + x^3\right) dx$$

$$= 2\pi\left[\left(x^2 - x^3 + \frac{x^4}{4}\right)\Big|_0^1\right] = \frac{2\pi}{4} = \frac{\pi}{2}$$