Learning objectives:

1. Understand the idea of how much work is done by a force.
2. Find work done by some variable force using integration.

The case of constant force

Work done is the product of force and displacement.

Example 1. How much work is done in lifting a 1.2 kg book off the floor to put it on a desk that is 0.7 m high? Take acceleration due to gravity to be $g = 10\text{m/s}^2$.

$$F = 1.2 \times g = 1.2 \times 10 = 12 \text{ N}$$

$\hookrightarrow$ kg·m/s²

$$s = 0.7 \text{ m}$$

$$W = F \cdot s = 12(0.7) = 8.4 \text{ Nm}$$

$$= 8.4 \text{ J}$$

$\hookrightarrow$ 1 Joule = 1 N·m

Example 2. A 360-lb gorilla climbs a tree to a height of 20 ft. Find the work done if the gorilla reaches the height in

1. 10 seconds,
2. 5 seconds.

In both the cases,

$$W = Fs = 360 \times 20 \text{ lb-ft} = 7200 \text{ lb-ft}$$

The case of variable force

Suppose an object moves along the x-axis under the influence of a force varying as $F(x)$. The work done in moving the object from $x = a$ to $x = b$ is given by

$$W = \int_a^b F(x)\,dx\,.$$

Example 3. When a particle is located at a distance of x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

$$W = \int_1^3 (x^2 + 2x)\,dx$$

$$= \left.\frac{x^3}{3}\right|_1^3 + \left. x^2 \right|_1^3 = \frac{3^3 - 1^3}{3} + 3^2 - 1^2$$

$$= \frac{26}{3} + 8 = \frac{50}{3} \text{ lb-ft.}$$

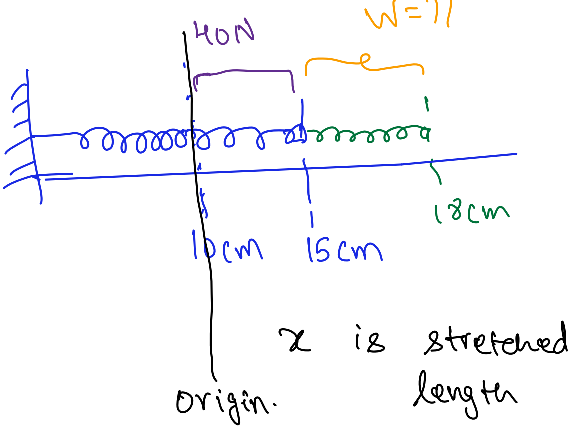
Hooke's law

The force required to maintain a spring stretched x units beyond its natural length is proportional to x. We have

$$F(x) = kx\,,$$

where k is a constant, called the spring constant.

Example 4. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?



$$F(x) = kx$$

$$F\left(\frac{5}{100}\right) = 40\text{N}$$

$$\Rightarrow k\left(\frac{5}{100}\right) = 40$$

$$\Rightarrow k = 800 \text{ N/m}$$

$$W = \int_{0.05}^{0.08} 800\,x\,dx = 800\,\frac{x^2}{2}\Big|_{0.05}^{0.08} = 400\left(0.08^2 - 0.05^2\right)$$

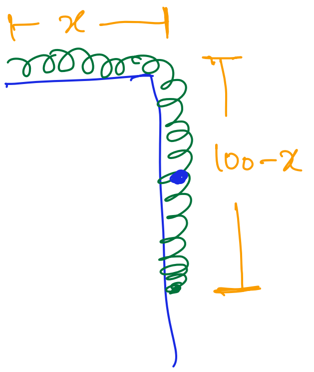
$$= 4\left(0.64 - 0.25\right)$$

$$= 1.56\ \text{J}$$

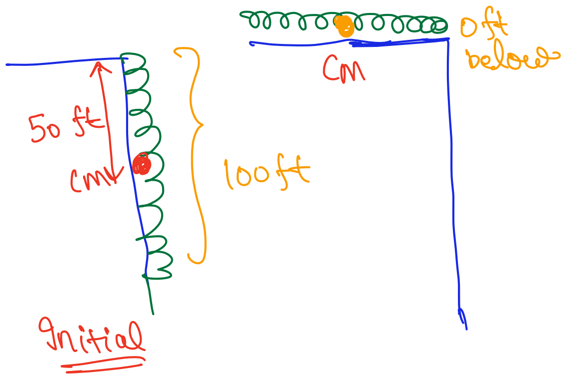
Example 5. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

λ = mass per unit length

$= \frac{200}{100}$ lb/ft $= 2$ lb/ft



Force needed to lift $(100-x)$ ft of cable is $(100-x)2$



The centre of mass moves from 50 ft. below to 0 ft below the top of the building.

$$\Rightarrow W = \int_0^{50} 2(100-x)\,dx$$

$$= 2\left(100x - \frac{x^2}{2}\right)\Big|_0^{50}$$

$$= 2\left(5000 - \frac{2500}{2}\right) = 7500\ \text{lb-ft.}$$