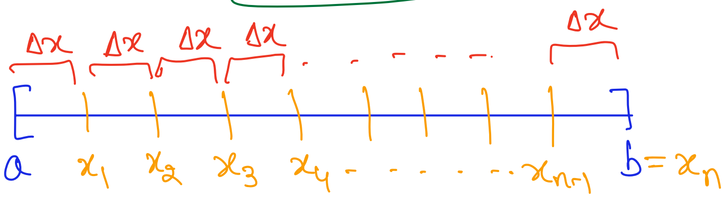
Learning objectives:

1. Find an expression for the average value of a function.
2. Understand the mean value theorem for integrals.

Average value of a function

Let f be a function defined on a closed interval $[a, b]$. Then the average value of f on the interval $[a, b]$ is given by

$$\boxed{f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx\,.}$$



$$f_{av} = \lim_{n\to\infty} \frac{f(x_1)+f(x_2)+\cdots\cdots+f(x_n)}{n}$$

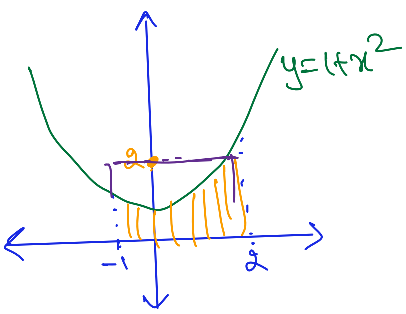
$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\underbrace{\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} f(x_i)}_{\int_a^b f(x)\,dx}$$

Example 1. Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$a=-1,\quad b=2,\quad f(x)=1+x^2$$

$$f_{av} = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx = \frac{1}{3}\left[\left(x+\frac{x^3}{3}\right)\Big|_{-1}^{2}\right]$$



$$= \frac{1}{3}\left[\{2-(-1)\}+\left\{\frac{2^3-(-1)^3}{3}\right\}\right]$$

$$= \frac{1}{3}\left[3+\frac{8-(-1)}{3}\right] = \frac{1}{3}\left[3+3\right] = 2$$

The mean value theorem for integrals. If f is continuous on $[a, b]$, then there exist a number c in $[a, b]$ such that

$$f(c) = f_{av} = \frac{1}{b-a} \int_a^b f(x)\,dx\,.$$

Example 2. Let $f(x) = 1 + x^2$ be as in Example 1. Find all possible numbers c for which $f(c) = f_{av}$.

$f_{av} = 2$

$f(c) = 2 \Rightarrow 1 + c^2 = 2$

$\Rightarrow c^2 = 1 \Rightarrow c = \pm 1 \rightarrow$ both lie in $[-1, 2]$