M16600 Lecture Notes

Section 10.2: Calculus with Parametric Curve

■ **Section 10.2** textbook exercises, *page 695*: #3, 4, 5, 7(a), 17, 11, 13. For #11, 13, only compute $\frac{d^2y}{dx^2}$, don't need to do concavity.

GOALS: Given a parametric curve $x = x(t)$ and $y = y(t)$

- Compute $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$
- Find the **slope of the tangent line** to the given parametric curve at a point.
- Write an **equation of the tangent line** to the given parametric curve at a point.
- Find points on parametric curves such that the tangent line is *horizontal* or *vertical*

Recall:

- Let $y = y(x)$ be a curve in the xy-plane (e.g $y = x^2 + 1$). Then

  the **SLOPE** of the TANGENT LINE to $y = y(x)$ at the point $x = a$ is $y'(a)$.

- The point-slope formula for **an equation of a line** is $y - y_1 = m(x - x_1)$ where (x_1, y_1) is one point on the line and m is the slope of the line.

Given a parametric curve: $x = x(t), y = y(t)$. We can compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$. How do we find $\frac{dy}{dx}$ so that we can compute the slope of a tangent line to this parametric curve?

Note that we can write $y(t)$ as the composite function $y(t) = y\big(x(t)\big)$, where $x(t)$ is the inner function. Then by the Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Therefore,

$$\boxed{\frac{dy}{dx} = \frac{y'(t)}{x'(t)}}$$

Geometrically, $\frac{dy}{dx}$ represents the ***slope formula*** of tangent lines to the parametric curve $x = x(t), y = y(t)$ at any point. To find the ***slope of the tangent line*** at one specific when $t = a$, we evaluate $\frac{dy}{dx}$ at $t = a$. Notation: $\left.\dfrac{dy}{dx}\right|_{t=a}$.

Given parametric equations $x = x(t), y = y(t)$, the second derivative of y with respect to x is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

would be in terms of t

Example 1: Let $x = t^2 - 3$ and $y = t^3 - 3t$. Find

(a) $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$

$$\frac{dx}{dt} = 2t$$

$$dy/dt = 3t^2 - 3$$

(b) $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t}$$

(c) the slope of the tangent line to the given parametric curve when $t = -2$

$$m \text{ at } t=-2 \text{ is } \left.\frac{dy}{dx}\right|_{t=-2} = \frac{3(-2)^2-3}{2(-2)} = -\frac{9}{4}$$

(d) an equation of the tangent line to the given parametric curve when $t = -2$

$$x_1 = x(-2) = (-2)^2 - 3 = 1$$

$$y_1 = y(-2) = (-2)^3 - 3(-2) = -8 + 6 = -2$$

$$y - (-2) = -\frac{9}{4}(x - 1) \Rightarrow y + 2 = -\frac{9x}{4} + \frac{9}{4}$$

$$\Rightarrow y = -\frac{9}{4}x + \frac{1}{4}$$

(e) an equation of the tangent line to the given parametric curve at the point $(-2, 2)$

Find t

$$t^2 - 3 = -2 \Rightarrow t^2 = 3 - 2 = 1 \Rightarrow t = \pm 1$$

$$t^3 - 3t = 2 \longrightarrow \text{Put } t = \pm 1 \text{ and check}$$

$\times$ $\boxed{t=1}$ $\quad 1^3 - 3 = -2 \neq 2$

✓ $\boxed{t=-1}$ $\quad (-1)^3 - 3(-1) = -1 + 3 = 2$

$\Rightarrow t = -1$

$$m_T = \left.\frac{dy}{dx}\right|_{t=-1} = \left.\frac{3t^2-3}{2t}\right|_{t=-1} = \frac{3(-1)^2-3}{2(-1)} = 0$$

$$y-2 = 0\,(x+2) \Rightarrow y-2=0 \Rightarrow \boxed{y=2}$$

$$y-y_1 = m\,(x-x_1)$$

$$y_1 = 2, \quad m = 0, \quad x_1 = -2$$

Example 2: Find an equation of the tangent line to the parametric curve

$$x = t - \sin t, \qquad y = 1 - \cos t$$

at $t = \pi/3$.

$$\frac{dx}{dt} = 1 - \cos t, \quad \frac{dy}{dt} = +\sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1-\cos t}$$

$$m_T = \left.\frac{dy}{dx}\right|_{t=\pi/3} = \frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1-1/2} = \sqrt{3}$$

$$x_1 = x(\pi/3) = \frac{\pi}{3} - \sin\frac{\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$$y_1 = y(\pi/3) = 1 - \cos\pi/3 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$y - y_1 = m_T\,(x - x_1)$$

$$\Rightarrow y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = \sqrt{3}\,x - \frac{\pi\sqrt{3}}{3} + \frac{3}{2}$$

$$\Rightarrow \boxed{y = \sqrt{3}\,x - \pi\frac{\sqrt{3}}{3} + 2}$$

Facts:

- The tangent line is ***horizontal*** at the values of t where $\frac{dy}{dx} = 0$.
- The tangent line is ***vertical*** at the values of t where $\frac{dy}{dx}$ is undefined.

Example 3: Let $\mathcal{C}$ be the parametric curve given by $x = t^3 - 3t$ and $y = t^3 - 3t^2$. Find

(a) Find the points on the curve $\mathcal{C}$ where the tangent line is horizontal.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 6t}{3t^2 - 3} = \frac{3(t^2 - 2t)}{3(t^2-1)} = \frac{t^2 - 2t}{t^2 - 1}$$

For horizontal $\frac{dy}{dx} = 0 \Rightarrow \frac{t^2 - 2t}{t^2 - 1} = 0 \Rightarrow t^2 - 2t = 0$

$$\Rightarrow t(t-2) = 0 \Rightarrow t = 0 \text{ or } t = 2$$

$t=0$ $t=2$

The required points $= (x(0), y(0))$ or $(x(2), y(2)) = (0,0), (2,-4)$

(b) Find the points on the curve $\mathcal{C}$ where the tangent line is vertical.

$\frac{dy}{dx} = \frac{t^2 - 2t}{t^2 - 1} =$ undefined (∞) $\Rightarrow t^2 - 1 = 0$

$\Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

$$x(t) = t^3 - 3t, \quad y(t) = t^3 - 3t^2$$

The required pts are $= (x(1), y(1))$ or $(x(-1), y(-1))$

$$= (-2, -2) \text{ or } (2, -4)$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

would be found in terms of t using $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Example 4: Let $x = 2t^3$ and $y = 2 + t^2$, find $\dfrac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{6t^2} = \frac{1}{3t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{1}{3t}\right)}{6t^2}$$

$$= \frac{\frac{1}{3}\left(-t^{-2}\right)}{6t^2} = \boxed{\frac{-1}{18t^4}}$$