M16600 Lecture Notes

Section 11.1: Sequences

■ **Section 11.1** textbook exercises, *page 744*: #3, 5, 13, 15, 23, 25, 27, 29, 31, 33, 35, 39, 41, 50.

Definition of a Sequence. A ***sequence*** is a set / collection of real numbers written in a definite order.

function $f: \mathbb{N} \to \mathbb{R}$

natural numbers $\{1, 2, 3, 4, \dots \infty\}$

E.g., $\{2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14, \dots, 2n, \dots\}$ is a sequence.

$f(1)$ $f(2)$ $f(3)$ . . . .

Find the 27^{th}-term of the above sequence.

$2(27) = 54$

Notation: A sequence $\{a_1,\ a_2,\ a_3,\ a_4, \dots, a_n, \dots\}$ could be written as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Sometimes we may start from a different integers

Note: n does not have to start from 1.

For the above sequence $\{2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14, \dots, 2n, \dots\}$,

$a_n = 2n$. Therefore, we could write this sequence as $\{2n\}_{n=1}^{\infty}$

Here are more examples of a sequences

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \dots\dots\right\}$$

$$\left\{\frac{(-1)^n}{n^2}\right\} = \left\{\frac{-1}{1}, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16}, \frac{-1}{25}, \frac{1}{36}, \frac{-1}{49}, \dots\dots\right\}$$

$$a_n = \frac{3^n}{(n+1)!} = \left\{\frac{3}{2}, \frac{9}{6}, \frac{27}{24}, \frac{81}{120}, \frac{243}{720}, \dots\dots\right\}$$

Here, for any positive integer k, $\quad k! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot k$.

$k!$ is read "k factorial"

$1! = 1$

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

$6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

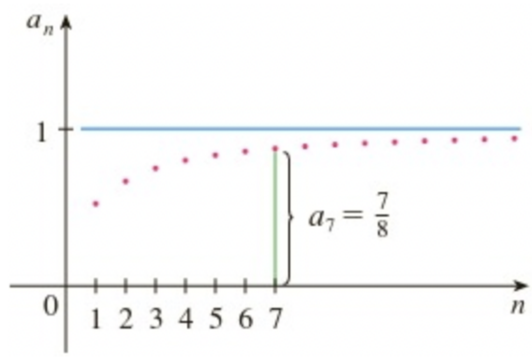
Example 1: Find a formula for the general term a_n of the sequence

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16} \cdots\right\}$$

$$a_n = \frac{n}{2^n}$$

Limit of a Sequence. We write $\lim_{n\to\infty} a_n = L$ if we can make the terms a_n as close to L as we like by taking n sufficiently large.

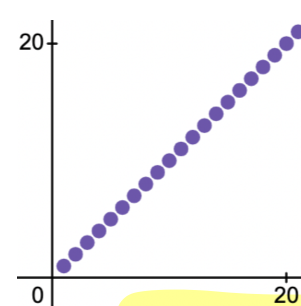
For example, given the sequence $a_n = \dfrac{n}{n+1}$, we have $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$ because the terms $a_n = \dfrac{n}{n+1}$ approaches 1 as n gets large. Below is the plot of some terms of this sequence.



Convergent or Divergent Sequence.

· If $\lim_{n\to\infty} a_n =$ (a finite number), then the sequence a_n is said to be ***convergent***.

· If $\lim_{n\to\infty} a_n = \pm\infty$ or $\lim_{n\to\infty} a_n$ does not exists , then the sequence a_n is said to be ***divergent***.



$a_n = n$
(divergent)

$a_n = (-1)^n$ is also

divergent

Example 2: Determine whether the sequence converges or diverges. If it converges, find the limit

$\lim_{n\to\infty} \frac{1}{n^k} = 0$ for any $k>0$

(a) $a_n = \dfrac{4n^2+2}{n+n^2}$. To answer this question, we want to compute $\lim\limits_{n\to\infty} a_n$.

Method 1 (an algebra approach): Factor as many x's as we can on the numerator and on the denominator then simplify. Then compute the limit.

$$\lim_{n\to\infty} \frac{4n^2+2}{n+n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(\frac{4\cancel{n^2}}{\cancel{n^2}} + \frac{2}{n^2}\right)}{\cancel{n^2}\left(\frac{n}{n^2} + \frac{\cancel{n^2}}{\cancel{n^2}}\right)} = \lim_{n\to\infty} \frac{4+\frac{2}{n^2}}{1/n+1} = \frac{4+2(0)}{0+1} = 4$$

Method 2 (a calculus approach): Use L'Hospital's Rule if applicable

$$\lim_{n\to\infty} \underbrace{\frac{4n^2+2}{n+n^2}}_{=\frac{\infty}{\infty}} = \lim_{n\to\infty} \underbrace{\frac{8n}{1+2n}}_{=\frac{\infty}{\infty}} = \lim_{n\to\infty} \frac{8}{2} = 4$$

Method 3 (the dropping-slower-terms approach): Keep the term with the largest growth rate of the numerator. Do the same for the denominator. Then simplify if possible. Then compute the limit.

Limit Facts:

$$\lim_{n\to\infty} \frac{\text{faster growth rate function}}{\text{slower growth rate function}} = \infty, \qquad \lim_{n\to\infty} \frac{\text{slower growth rate function}}{\text{faster growth rate function}} = 0$$

Within algebraic functions:

- n^a grows faster than n^b if $a>b$
- n^b grows slower that n^a if $b<a$

$$\lim_{n\to\infty} \frac{4n^2+2}{n+n^2}$$

$$= \lim_{n\to\infty} \frac{4\cancel{n^2}}{\cancel{n^2}} = 4$$

(b) $\left\{\dfrac{3\sqrt{n}}{n+\sqrt[3]{n^2}-5}\right\}$

$$\Rightarrow \lim_{n\to\infty} \frac{3\, n^{1/2}}{n+n^{2/3}-5} = \lim_{n\to\infty} \frac{3n^{1/2}}{n}$$

$$= \lim_{n\to\infty} \frac{\text{slower}}{\text{faster}} = \boxed{0}$$

(c) $a_n = \dfrac{\sqrt{10+n+3n^2+4n^5}}{6n^2+2n}$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{4n^5}}{6n^2} = \lim_{n\to\infty} \frac{2\, n^{5/2}}{6n^2}$$

$\left(\frac{5}{2} > 2\right)$

$$= \lim_{n\to\infty} \frac{\text{faster}}{\text{slower}} = \boxed{\infty}$$

(d) $a_n = e^{-2/n^2}$

$$\lim_{n\to\infty} e^{-2/n^2} = e^{\lim_{n\to\infty} \frac{-2}{n^2}} = e^0 = \boxed{1}$$

$$\lim_{n\to\infty} \frac{-2}{n^2} = \lim_{n\to\infty} \frac{\text{slower}}{\text{faster}} = 0$$

THE GROWTH RATE ORDER OF DIFFERENT TYPES OF FUNCTIONS.

logarithmic functions << algebra << exponential functions << factorial

Example 3: Determine whether the sequence converges or diverges. If it converges, find the limit

- algebraic fns. $\rightarrow n^k$ $(k>0)$
- within exponential fns. $a^x << b^x$ if $a<b$

(a) $\left\{\dfrac{\ln n}{n}\right\}$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\text{slower}}{\text{faster}} = 0$$

(b) $\left\{\dfrac{2^n}{5^n+4}\right\}$

$$\lim_{n\to\infty} \frac{2^n}{5^n} = \lim_{n\to\infty} \frac{\text{slower}}{\text{faster}} = 0$$

(c) $a_n = n!e^{-2n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{e^{2n}} = \lim_{n\to\infty} \frac{\text{faster}}{\text{slower}} = \infty$$