M16600 Lecture Notes

Section 11.10: Taylor and Maclaurin Series

■ **Section 11.10** textbook exercises, *page 811*: #6, 8, 9, 19, 21, 23, 25, 35, 37, 54. For #54, use the series representation for $\sin x$ in Table 1, page 808.

Taylor Series is a power series with a formula for the coefficient c_n. How do we find the formula for the coefficients? We will start out with the general form for power series

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-2)^3 + c_4(x-a)^4 + \cdots,$$

then compute $f(a)$, $f'(a)$, $f''(a)$, $f'''(a)$, *etc.* and see if we can find a pattern for c_n:

$$f(a) = C_0 + \underbrace{C_1(0) + C_2(0)^2 + C_3(0)^3 + \cdots}_{=0}$$

$$\Rightarrow f(a) = C_0$$

$$f'(x) = C_1 + \underbrace{2C_2(x-a) + 3C_3(x-a)^2 + 4C_4(x-a)^3 + \cdots}_{=0 \text{ at } x=a}$$

$$\Rightarrow f'(a) = C_1$$

$$f''(x) = 2C_2 + \underbrace{3\cdot 2\, C_3(x-a) + 4\cdot 3\, C_4(x-a)^2 + 5\cdot 4\, C_5(x-a)^3 + \cdots}_{=0 \text{ at } x=a}$$

$$\Rightarrow f''(a) = 2C_2$$

$$f'''(x) = 3\cdot 2\cdot C_3 + \underbrace{4\cdot 3\cdot 2\, C_4(x-a) + 5\cdot 4\cdot 3\, C_5(x-a)^2 + \cdots}_{=0 \text{ at } x=a}$$

$$\Rightarrow f'''(a) = 6C_3$$

In general, $f^{(n)}(a) = n!\, C_n$

TAYLOR SERIES OF $f(x)$ AT a.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

↳ completely determined from f

$$\Rightarrow C_n = \frac{f^{(n)}(a)}{n!}$$

A special case of Taylor series is when the center $a = 0$. This special is given a name called ***Maclaurin series***.

MACLAURIN SERIES (TAYLOR SERIES CENTERED AT 0). $$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

Example 1: Use the definition of Taylor series to find the first four nonzero terms of the series for $f(x) = \ln x$ centered at $a = 1$.

$f(x) = \ln x \Rightarrow f(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$

$f^{(4)}(x) = \frac{-6}{x^4} \Rightarrow f^{(4)}(1) = -6$

$C_0 = f(1) = \ln(1) = 0$

$C_1 = f'(1) = \frac{1}{1} = 1$

$C_2 = \frac{f''(1)}{2!} = \frac{-1}{2}$

$C_3 = \frac{f'''(1)}{3!} = \frac{2}{6} = \frac{1}{3}$

$C_4 = \frac{f^{(4)}(1)}{4!} = \frac{-6}{24} = \frac{-1}{4}$

$$\Rightarrow \ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

Example 2: Find the Taylor series for $f(x) = \dfrac{1}{1+x}$ centered at $a = 2$.

$f(x) = \frac{1}{1+x} \Rightarrow f(2) = \frac{1}{1+2} = \frac{1}{3} \Rightarrow C_0 = f(2) = \frac{1}{3}$

$f'(x) = \frac{-1}{(1+x)^2} \Rightarrow f'(2) = \frac{-1}{(1+2)^2} = \frac{-1}{9} \Rightarrow C_1 = \frac{f'(2)}{1!} = \frac{-1}{9}$

$f''(x) = \frac{2}{(1+x)^3} \Rightarrow f''(2) = \frac{2}{(1+2)^3} = \frac{2}{27} \Rightarrow C_2 = \frac{f''(2)}{2!} = \frac{1}{2} \cdot \frac{2}{27} = \frac{1}{27}$

$f'''(x) = \frac{-6}{(1+x)^4} \Rightarrow f'''(2) = \frac{-6}{(1+2)^4} = \frac{-6}{81} \Rightarrow C_3 = \frac{f'''(2)}{3!} = \frac{1}{6} \cdot \frac{-6}{81} = \frac{-1}{81}$

In general $C_n = (-1)^n \left(\frac{1}{3}\right)^{n+1} = \frac{(-1)^n}{3^{n+1}}$

$$\frac{1}{1+x} = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3 \pm \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-2)^n$$

Example 3: Use the definition of Maclaurin series to find the Maclaurin series of $f(x) = e^x$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad (a=0)$$

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1 \Rightarrow c_0 = \frac{f(0)}{0!} = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1 \Rightarrow c_1 = \frac{f'(0)}{1!} = 1$$

$$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1 \Rightarrow c_2 = \frac{f''(0)}{2!} = \frac{1}{2}$$

$$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1 \Rightarrow c_3 = \frac{f'''(0)}{3!} = \frac{1}{6}$$

In general, $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1 \Rightarrow c_n = \frac{1}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots$$

Example 4: Use the result in example 3 to find the Maclaurin series for

(a) $f(x) = e^{-x^2}$

Replace x with $-x^2$ in the Maclaurin series for e^x

$$\Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

(b) $f(x) = xe^x$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

multiply both sides by x

$$x e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \cdot x = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}$$

Example 5: (a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series. (Note, we cannot compute this indefinite integral using any of the integral techniques we've learned in chapter 7)

$$\Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

$$\int e^{-x^2}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$

(b) Evaluate $\int_0^1 e^{-x^2}\,dx$ using the first four terms of the power series you found in part (a).

$$\int_0^x e^{-t^2}\,dt = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{t^{2n+1}}{2n+1}\Bigg|_0^x$$

$$= C + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}}_{=0 \text{ when } x=0}$$

$$x=0 \Rightarrow \int_0^0 e^{-t^2}\,dt = 0 = C$$

$$\Rightarrow C = 0$$

Put $x=1$ $\Rightarrow$ $$\int_0^1 e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2n+1}$$

$$= 1 - \frac{(-1)^1}{1!}\frac{1}{3} + \frac{(-1)^2}{2!}\frac{1}{5} + \frac{(-1)^3}{3!}\frac{1}{7} + \cdots \; 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42}$$

$$= 0.67 + 0.1 - 0.024$$

$$= 0.746$$