M16600 Lecture Notes
Section 11.2: Series

■ **Section 11.2** textbook exercises, *page 755*: #6, 15, 22, 23, 24, 26, 29, 31, 33, 37, 46, 47.

DEFINITION OF SERIES. An ***infinite series*** (or just ***series***) is an infinite SUM of the terms of the sequence $\{a_n\}$

Series Notation:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

Note: n does not have to start from 1.

E.g., $\displaystyle\sum_{n=1}^{\infty} 2^n = \underbrace{2 + 4 + 8 + 16 + 32 + 64 + 128 + \cdots}_{\text{infinitely many terms.}}$

Here, $a_n = 2^n$

PARTIAL SUMS OF A SERIES. If we have a series $\displaystyle\sum_{n=1}^{\infty} a_n$ then

· the first partial sum $s_1 = a_1$

· the second partial sum $s_2 = a_1 + a_2$

· the 3^{rd} partial sum $s_3 = a_1 + a_2 + a_3$

· the n^{th} partial sum $s_n = a_1 + a_2 + \cdots\cdots + a_n$ (sum of first n terms)

Example 1: Find the 4^{th} partial sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$s_4 = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

DEFINITION OF CONVERGENT AND DIVERGENT SERIES. Given a series $\displaystyle\sum_{n=1}^{\infty} a_n$, we can establish a sequence of its *partial sums* $\{s_n\} = \{\ s_1,\ s_2,\ s_3, \ldots, s_n, \ldots\ \}$

We can compute $\displaystyle\lim_{n\to\infty} s_n$. If

$\displaystyle\lim_{n\to\infty} s_n \longrightarrow$ value of the series / infinite sum $\displaystyle\sum_{n=1}^{\infty} a_n$

$$\begin{cases} \displaystyle\lim_{n\to\infty} s_n = \pm\infty, & \text{then } \displaystyle\sum_{n=1}^{\infty} a_n \text{ is } \textbf{divergent} \\ \displaystyle\lim_{n\to\infty} s_n = S, \text{ a finite number}, & \text{then } \displaystyle\sum_{n=1}^{\infty} a_n \text{ is } \textbf{convergent} \text{ and } \displaystyle\sum_{n=1}^{\infty} a_n = S \end{cases}$$

Remark: By writitng $\sum_{n=1}^{\infty} a_n = S$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number S.

Example 2: Given the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Calculate the first eight terms of the sequence of partial sums correct to the four decimal places. Does it appear that the series is convergent or divergent?

$$s_1 = \frac{1}{2} = 0.5$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = 0.5 + 0.25 = 0.75$$

$$s_3 = \left(\frac{1}{2} + \frac{1}{4}\right) + \frac{1}{8} = 0.75 + 0.125 = 0.875$$

$$s_4 = s_3 + \frac{1}{16} = 0.875 + 0.0625 = 0.9375$$

$$s_5 = s_4 + \frac{1}{32} = 0.9375 + 0.03125 = 0.96875$$

$$s_6 = s_5 + \frac{1}{64} = 0.96875 + 0.015625$$

$$\simeq 0.9688 + 0.0156 = 0.9844$$

$$\vdots$$

Series with Names. There are three special series which come up fairly often in Chapter 11.

- ***Geometric Series:***

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

r is called the **common ratio** of the geometric series.

Remark: For a GEOMETRIC series, the first term is always a and the second term is always ar.   The third term is ar^2.

E.g., $\sum_{n=1}^{\infty} \frac{2}{3^n}$ is a geometric series. Find a and r for this geometric series.

$a_n = \frac{2}{3^n} \Rightarrow$ Put $n=1$, $a = \frac{2}{3^1} = \frac{2}{3}$

Put $n=2$, $ar = \frac{2}{3^2} = \frac{2}{9}$

$$r = \frac{ar}{a} = \frac{2/9}{2/3} = \frac{2}{9} \cdot \frac{3}{2} = \frac{1}{3}$$

Convergence/Divergence Test for a Geometric Series.

$$\begin{cases} \text{The geometric series } \sum_{n=1}^{\infty} ar^{n-1} \text{ is } \textbf{divergent} \text{ if } & |r| \geq 1 \\ \text{The geometric series } \sum_{n=1}^{\infty} ar^{n-1} \text{ is } \textbf{convergent} \text{ if } & |r| < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \end{cases}$$

Example 3: Is the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergent or divergent? If it converges, find its sum

$a_n = \frac{1}{2^n} \Rightarrow$ Put $n=1$, $\left. a = \frac{1}{2} \right\} a$

Put $n=2$, $ar = \frac{1}{2^2} = \frac{1}{4}$

$$r = \frac{ar}{a} = \frac{1/4}{1/2} = \frac{1}{4} \cdot 2 = \underbrace{\frac{1}{2}}_{r}$$

$|r| = \frac{1}{2} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$$

- ***The p-Series:*** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a real number. (section 11.3)

Convergence/Divergence Test for a p-Series.

$$\begin{cases} \text{The } p\text{-series } \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \textbf{divergent} \text{ if } & p \leq 1 \\ \text{The } p\text{-series } \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \textbf{convergent} \text{ if } & p > 1. \end{cases}$$

Here are examples of p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

$\hookrightarrow P = 3$

$\Rightarrow$ convergent

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$\hookrightarrow P = 1/2$

$\Rightarrow$ divergent

Examples

① $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \to p = \frac{1}{2}$ (div)

② $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \to p = 1$ (div)

③ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \to p = 2$ (conv)

④ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \to p = 5/2$ (conv)

⑤ $\displaystyle\sum_{n=1}^{\infty} \sqrt{n} \to p = -1/2$ (div)

$\hookrightarrow \dfrac{1}{1/\sqrt{n}} = \dfrac{1}{n^{-1/2}}$

- ***Telescoping Series:***

An example of a telescoping series is $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

There is no quick test of convergence/divergence of telescoping series. To test the **Convergence/Divergence for Telescoping Series**, we must use the **definition of convergent and divergent series** on page 1.

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

$$\boxed{\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= 0 - 0 = 0\\ &\Rightarrow \text{TD is inconclusive}\end{aligned}}$$

$$s_1 = a_1 = \frac{1}{1} - \frac{1}{2}$$

$$s_2 = a_1 + a_2 = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$s_3 = \underbrace{a_1 + a_2}_{s_2} + a_3 = 1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} = 1 - \frac{1}{4}$$

$$s_4 = s_3 + a_4 = 1 - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \frac{1}{5} = 1 - \frac{1}{5}$$

$$s_n = a_1 + a_2 + \cdots + a_n$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - \lim_{n\to\infty}\frac{1}{n+1}$$

$$= 1$$

Here is a very useful tool to see whether a series is **divergent**

Test for Divergence (TD). Given a series $\sum a_n$. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

i.e. if $\sum_{n=1}^{\infty} a_n$ is convergent then we must have $\lim_{n\to\infty} a_n = 0$

Example 4: Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$a_n = \frac{n^2}{5n^2+4}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{\cancel{n^2}}{5\cancel{n^2}} = \frac{1}{5} \neq 0$$

$\Rightarrow$ By TD, $\sum_{n=1}^{\infty} a_n$ diverges.

Warning: If $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ could be convergent or divergent. We don't know! **Never** conclude that a series is convergent if you use the Test for Divergence.

Example 5: Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

Note: We know a series is a geometric series if the term a_n can be rewritten as $(\text{constant})(r)^{\text{exponent in terms of } n}$.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ $\longrightarrow$ $a = a_1 = \dfrac{(-3)^{1-1}}{4^1} = \dfrac{1}{4}$

$$a_n = \frac{(-3)^{n-1}}{4^n} \Rightarrow a_{n+1} = \frac{(-3)^{n+1-1}}{4^{n+1}} = \frac{(-3)^n}{4^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n}{4^{n+1}} \cdot \frac{4^n}{(-3)^{n-1}} = (-3)^{n-(n-1)}\, 4^{n-(n+1)}$$

$$= (-3)\, 4^{-1} = -\frac{3}{4}$$

$\Rightarrow$ series is geometric with $r = -\frac{3}{4}$

$|r| = \frac{3}{4} < 1 \Rightarrow$ convergent and $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1/4}{1-(-3/4)} = \frac{1/4}{1+3/4} = \frac{1}{7}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n+1}}{(-2)^n}$

Find the first term: $a = \dfrac{3^{2(0)+1}}{(-2)^0} = \dfrac{3}{1} = 3$
(Put $n=0$)

$$a_n = \frac{3^{2n+1}}{(-2)^n} \Rightarrow a_{n+1} = \frac{3^{2(n+1)+1}}{(-2)^{n+1}} = \frac{3^{2n+2+1}}{(-2)^{n+1}} = \frac{3^{2n+3}}{(-2)^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{2n+3}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{3^{2n+1}} = 3^{2n+3-(2n+1)} \cdot (-2)^{n-(n+1)}$$

$$= 3^{\cancel{2n}+3-\cancel{2n}-1} \cdot (-2)^{\cancel{n}-\cancel{n}-1} = \frac{3^2}{-2} = \frac{-9}{2}$$

$\Rightarrow r = -\dfrac{9}{2} \Rightarrow |r| = \dfrac{9}{2} > 1 \Rightarrow$ The geometric series is divergent

Example 6: Determine whether the series is convergent or divergent.

Hint: Determine whether each series is a geometric series or a p-series first. If a series is neither one of those, think about using the Test of Divergence.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3+1}{k^2+2k+5}$

$\longrightarrow$ Not a geometric series

$\longrightarrow$ Not a p-series.

$$\lim_{k\to\infty} \frac{k^3+1}{k^2+2k+5} = \lim_{k\to\infty} \frac{k^3}{k^2} = \lim_{k\to\infty} \frac{\text{faster}}{\text{slower}} = \infty \neq 0$$

By TD, given series is divergent

(b) $\sum_{n=1}^{\infty} 4^{-n}3^{n+1}$ $\longrightarrow$ First term : $a = 4^{-1}3^{1+1} = \frac{9}{4}$

$a_n = 4^{-n}3^{n+1} \Rightarrow a_{n+1} = 4^{-(n+1)}3^{n+1+1} = 4^{-n-1}3^{n+2}$

$\frac{a_{n+1}}{a_n} = \frac{4^{-n-1}3^{n+2}}{4^{-n}3^{n+1}} = 4^{-n-1-(-n)}3^{n+2-(n+1)}$

$= 4^{\cancel{-n}-1\cancel{+n}}3^{\cancel{n}+2\cancel{-n}-1} = 4^{-1}\cdot 3^{1} = \frac{3}{4}$

$r = \frac{3}{4} \Rightarrow |r|<1 \Rightarrow$ convergent and $\sum_{n=1}^{\infty} 4^{-n}3^{n+1} = \frac{9/4}{1-\frac{3}{4}} = \frac{9/4}{1/4}$

$= 9$

(c) $\sum_{n=1}^{\infty} \frac{1}{e^{-n}+2}$

$\longrightarrow$ not a P-series

$\longrightarrow$ not a geometric series

(not a Product/quotient of $\underbrace{\text{exponential fns}}_{(b^n)}$)

$\Downarrow$

use TD

$\lim_{n\to\infty} \frac{1}{e^{-n}+2} = \frac{1}{\lim_{n\to\infty} e^{-n}+2} = \frac{1}{0+2} = \frac{1}{2} \neq 0$

$\Rightarrow$ By TD, given series diverges

(d) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

P-series with $P=2>1$

$\Rightarrow$ given series is convergent.