M16600 Lecture Notes

Section 11.3: The Integral Test

■ **Section 11.3** textbook exercises, *page 765*: #3, 5, 7, 21, 23, 22. **Note:** For # 21, 23, 22, show that the conditions of the Integral Test are true.

THE INTEGRAL TEST. Suppose f is a *continuous, positive, decreasing* function on $[1, \infty)$ and let $a_n = f(n)$. Then

(i) If $\displaystyle\int_1^\infty f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^\infty f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Note: When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_4^\infty \frac{1}{(n-3)^2}\,dx$$

Also, it is not necessary that f be *always* decreasing. What is important is that f be *ultimately* decreasing, that is decreasing for x larger than some number N.

Example 1: Use the Integral Test to test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Show that the conditions of the Integral Test are true for this problem.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} \xrightarrow{f(x)=?} a_n = \frac{1}{n^2+1}$ (Replace n with x) $\Rightarrow f(x) = \dfrac{1}{x^2+1}$

- Is $f(x)$ continuous on $[1, \infty)$? Yes. (denominator is not zero and made up of cont. fns

- Is $f(x)$ positive on $[1, \infty)$? Yes.

  $x^2+1 > 0$ always. $\Rightarrow \dfrac{1}{x^2+1} > 0$

- Is $f(x)$ ultimately decreasing? Yes.

  $f'(x) = \dfrac{-1}{(x^2+1)^2} \cdot (2x) = \dfrac{-2x}{(x^2+1)^2} < 0$ for $x > 0$

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan(x)\Big|_1^t$$

$$= \lim_{t\to\infty} \arctan(t) - \arctan(1) = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$$ $\Rightarrow$ Integral converges

$\Rightarrow$ given series converges

Example 2: Use the Integral Test to test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergence. Show that the conditions of the Integral Test are true for this problem.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} \Rightarrow a_n = \frac{\ln(n)}{n} \Rightarrow f(x) = \frac{\ln x}{x}$$

- Is $f(x)$ continuous on $[1,\infty)$? Yes.

  $\ln x$ is cont, x is cont $\Rightarrow \frac{\ln x}{x}$ is cont. since the denominator is not zero. on $[1,\infty)$

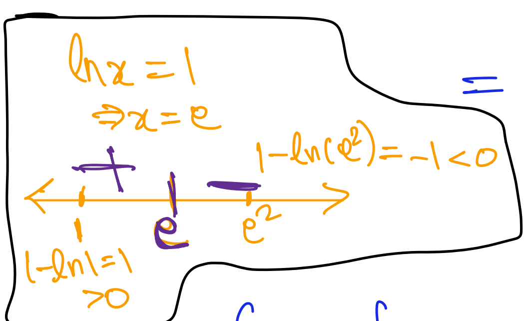
- Is $f(x)$ +ve on $[1,\infty)$? Yes.

  $\ln x \geq 0$ for $x \geq 1$, and $x \geq 1$

  $\Rightarrow \frac{\ln x}{x} \geq 0$ on $[1,\infty)$

- Is $f(x)$ ultimately decreasing? Yes

$$f'(x) = \frac{x\cdot(\ln x)' - (x)'\cdot \ln x}{x^2} = \frac{x\cdot\frac{1}{x} - 1\cdot \ln x}{x^2}$$



$$= \frac{1-\ln x}{x^2}$$ $\longrightarrow$ $1-\ln x < 0 \Rightarrow 1 < \ln x$

$x^2 \longrightarrow$ always +ve

$\Rightarrow \ln x > 1$

$\Rightarrow x > e$

So, for $x > e$, $f'(x) < 0$.

$\Rightarrow$ we find convergence|divergence of $\int_1^\infty \frac{\ln x}{x}\,dx$

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx$$

$$\int_1^t \frac{\ln x}{x}\,dx = \int_{\ln 1}^{\ln t} u\,du = \frac{u^2}{2}\Big|_0^{\ln t}$$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$$= \frac{1}{2}(\ln t)^2$$

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 = \infty$$

$\Rightarrow \int_1^\infty \frac{\ln x}{x}\,dx$ diverges

Hence, the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ also diverges.