**M16600 Lecture Notes**

*Section 11.4: The Comparison Tests*

■ **Section 11.4** textbook exercises, *page 771*: #3, 5, 7, 10, 12, 13, 15, 19, 23, 25, 29.

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In the comparison tests the idea is to compare the given series with a series that is known to be convergent or divergent.

For instance, say we would like to determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent or divergent. Note that the Test of Divergence fails for this series and this is not a geometric series or a $p$-series.

$$\lim_{n\to\infty} \frac{1}{2^n+1} = \lim_{n\to\infty} \frac{1}{2^n} = \lim_{n\to\infty} \frac{\text{slower}}{\text{faster}} = 0 \Rightarrow \text{TD does not apply}$$

On the other hand, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ reminds us of the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ with $r = \frac{1}{2}$; hence, the latter series is convergent. We can do the following comparison between these two series to determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.

$$2^n + 1 > 2^n \Rightarrow \frac{1}{2^n+1} < \frac{1}{2^n}$$

$$\boxed{\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1}$$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series.

$$r = \frac{1}{2^{n+1}} \cdot 2^n = \frac{1}{2} \quad \text{and} \quad a = \frac{1}{2^1} = \frac{1}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} \text{ is convergent.}$$

**The Comparison Test.** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for large enough $n$, then $\sum a_n$ is also Convergent.
(ultimately)

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for large enough $n$, then $\sum a_n$ is also divergent.

**Remark:** The Comparison Test is useful when testing series with sine or cosine functions.

*Example 1:* Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{7^n}$ converges of diverges.

$$-1 \le \sin(n) \le 1 \Rightarrow 0 \le 1+\sin(n) \le 2$$

$$\Rightarrow 0 \le \frac{1+\sin(n)}{7^n} \le \frac{2}{7^n}$$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1+\sin(n)}{7^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{2}{7^n}$ are both having +ve terms

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{7^n}$ is convergent? $a_n = \frac{2}{7^n}$, $a_{n+1} = \frac{2}{7^{n+1}}$

$\Rightarrow \dfrac{a_{n+1}}{a_n} = \dfrac{2}{7^{n+1}} \cdot \dfrac{7^n}{2} = \dfrac{1}{7} < 1 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{2}{7^n}$ is convergent by the geometric series test

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1+\sin(n)}{7^n}$ is also convergent (by CT)

*Question:* Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 3}$ convergent or divergent?

$2^n - 3 < 2^n \Rightarrow \dfrac{1}{2^n-3} > \dfrac{1}{2^n} \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-3} > \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

$\hookrightarrow$ cannot say conv./div. from CT

$2^n - 3 > r^n$ for some number $r$. Take $r = 1.5$

$2^n - 3 > \left(\frac{3}{2}\right)^n$, $n \geq 3$ $\Rightarrow \displaystyle\sum_{n=3}^{\infty} \frac{1}{2^n-3} \boxed{<} \sum_{n=3}^{\infty} \frac{1}{(3/2)^n} = \underbrace{\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n}$

Conv. $\left\{ \begin{array}{l} \text{geometric series} \\ \text{with } r = 2/3 \end{array} \right.$

$\Rightarrow \displaystyle\sum_{n=3}^{\infty} \frac{1}{2^n-3}$ also converges.

$\boxed{\hookrightarrow}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-3}$ also converges.

The ***Limit Comparison Test*** helps us to determine the convergence or divergence of a series that is "similar" to a series which we're familiar with.

**DEFINITION OF SIMILARITY BETWEEN TWO SERIES.** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \text{ a positive number },$$ (cannot be 0 or $\infty$)

then we say $\sum a_n$ and $\sum b_n$ are ***similar to*** each other.

**THE LIMIT COMPARISON TEST:** Suppose $\sum a_n$ and $\sum b_n$ are *similar series* with positive terms. Then **either** both series are convergent **or** both series are divergent.

In other words, similar series behave the same way regarding convergence or divergence.

*Example 2:* Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4}$ is similar to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$. Then use the Limit Comparison Test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4}$ is convergent or divergent.

$$a_n = \frac{1}{\sqrt{n}+4}, \qquad b_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+4} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 1$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4} \backsim \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ is divergent (p-series with $p=\frac{1}{2}$)

$\Rightarrow$ By LCT, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4}$ is also divergent

**Remark:** The Limit Comparison Test is very useful when working with series that remind us of geometric series or $p$-series.

**Remark:** To determine similar series, often we can drop the slower terms of the numerator and of the denominator then use algebra to simplify.

*Example 3:* Find the similar series of the given series then test for convergence and divergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2} \backsim \sum_{n=1}^{\infty} \frac{n^2}{n^4} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}$

Converges since its a P-series with $P=2$

By the LCT, given series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$ also converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n+n}{5^n-2} \backsim \sum_{n=1}^{\infty} \frac{6^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ (geometric series)

$$r = \frac{a_{n+1}}{a_n} = \frac{(6/5)^{n+1}}{(6/5)^n} = \frac{6}{5} > 1$$

$\Rightarrow$ the geometric series diverges.

By L.C.T., given series $\displaystyle\sum_{n=1}^{\infty} \frac{6^n+n}{5^n-2}$ also diverges.

*Example 4:* Determine whether the series converges of diverges.

(a) $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n+9}}$ $\backsim \sum_{n=1}^{\infty} \frac{5}{\sqrt{n}} = 5 \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$\hookrightarrow$ P-series with $P = \frac{1}{2} < 1$

$\Rightarrow$ It diverges

By LCT, $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n+9}}$ also diverges.

(b) $\sum_{n=1}^{\infty} \frac{2n(11+n)^{12}}{(8+3n)(1+n^2)^3}$ $\backsim \sum_{n=1}^{\infty} \frac{2n\,(n)^{12}}{3n\,(n^2)^3} = \frac{2}{3} \sum_{n=1}^{\infty} \frac{n^{13}}{n^7}$

$$= \frac{2}{3} \sum_{n=1}^{\infty} n^6 \qquad n^6 = \frac{1}{n^{-6}}$$

$\Rightarrow$ P-series with $P = -6$

So, it diverges.

$\Rightarrow$ By LCT, $\sum_{n=1}^{\infty} \frac{2n(11+n)^{12}}{(8+3n)(1+n^2)^3}$ also diverges.

(c) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{e^n + 3}$

$$0 \le \cos^2(n) \le 1 \Rightarrow \frac{\cos^2(n)}{e^n+3} \le \frac{1}{e^n+3}$$

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{e^n+3} \le \sum_{n=1}^{\infty} \frac{1}{e^n+3} \backsim \sum_{n=1}^{\infty} \frac{1}{e^n} = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n}_{\text{geometric series}}$$

$$r = \frac{(1/e)^{n+1}}{(1/e)^n} = \frac{1}{e} < 1$$

$\Rightarrow$ It converges.

By LCT, this converges

By CT, given series converges.

(d) $\sum_{n=1}^{\infty} \frac{n}{e^n}$ (use integral test) $f(x) = \frac{x}{e^x} = xe^{-x}$

- $f$ is cont. on $[1,\infty)$ because both $x$ and $e^x$ are cont. and $e^x \neq 0$ for any $x \geq 1$
- $f$ is +ve on $[1,\infty)$ because $x \geq 1 > 0$ and $e^x > 0$
- $f$ is ultimately decreasing : $f'(x) = e^{-x} - xe^{-x} = \underbrace{e^{-x}}_{\text{always +ve}}\underbrace{(1-x)}_{\text{-ve for } x \geq 1}$

  because $f'(x) < 0$ when $x \geq 1$

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x}\,dx$$

$$\int \underbrace{x}_{u} \underbrace{e^{-x}\,dx}_{dv} = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}$$

$u = x \Rightarrow du = dx$

$dv = e^{-x}dx$

$v = -e^{-x}$

$$= \lim_{t\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \left(-te^{-t} - e^{-t}\right) - \left(-e^{-1} - e^{-1}\right)$$

$$= \left(-\lim_{t\to\infty} \frac{t}{e^t} - \lim_{t\to\infty} e^{-t}\right) + 2e^{-1}$$

L'H Rule

$$= \left(-\lim_{t\to\infty} \frac{1}{e^t} - 0\right) + 2e^{-1} = (-0-0) + 2e^{-1} = 2e^{-1}$$

$$\Rightarrow \int_1^{\infty} \frac{x}{e^x}\,dx \quad \text{converges}$$

So, by the integral test, $\sum_{n=1}^{\infty} \frac{n}{e^n}$ also converges.