■ **Section 11.5** textbook exercises, *page 776*: # 4, 5, 7, 9, 6, 14.

Definition. An ***alternating series*** is a series whose terms are alternately positive and negative.

E.g., $$\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1}}_{\text{sign}} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \pm \cdots$$

As a convention, we write an alternating series as $\sum (-1)^n b_n$, where $b_n > 0$ for all n.

For the example above, $b_n = \frac{1}{n}$

Convergence/Divergence for Alternating Series $\sum (-1)^n b_n$

- **Alternating Series Test (AST):** The alternating series $\sum (-1)^n b_n$ **converges** if these two conditions are satisfied:

  (i) $\lim_{n\to\infty} b_n = 0$

  (ii) $b_{n+1} \leq b_n$ (the terms b_n are decreasing)

- The alternating series $\sum (-1)^n b_n$ **diverges** if $\lim_{n\to\infty} b_n \neq 0$.

Example 1: Use the Alternating Series Test to show that the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges.

$b_n = \frac{1}{n}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Check: $b_{n+1} \leq b_n$

$\frac{1}{n+1} \leq \frac{1}{n} \longrightarrow$ True because $n+1 > n$

$\frac{1}{n+1} < \frac{1}{n}$

$\Rightarrow$ By AST, given series converges.

Example 2: Test the series for convergence or divergence

Hint: The first step in determining convergence or divergence for an **alternating series** is to compute $\lim_{n\to\infty} b_n = 0$.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}+5}$ $\Rightarrow b_n = \frac{1}{2\sqrt{n}+5}$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2\sqrt{n}+5} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = \lim_{n\to\infty} \frac{\text{slower}}{\text{faster}} = 0$$

$$b_{n+1} = \frac{1}{2\sqrt{n+1}+5} \quad \because \quad n+1 > n \Rightarrow \sqrt{n+1} > \sqrt{n}$$

$$\Rightarrow 2\sqrt{n+1} > 2\sqrt{n}$$

$$\Rightarrow 2\sqrt{n+1}+5 > 2\sqrt{n}+5$$

$$\Rightarrow b_{n+1} < b_n$$

$$\Rightarrow \frac{1}{2\sqrt{n+1}+5} < \frac{1}{2\sqrt{n}+5}$$

By AST, the given series converges.

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n^4+n}{4n^4-n^3+n^2+2}$

$$b_n = \frac{3n^4+n}{4n^4-n^3+n^2+2}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3\cancel{n^4}}{4\cancel{n^4}} = \frac{3}{4} \neq 0$$

$\Rightarrow$ The given series diverges by the AST.