M16600 Lecture Notes

Section 11.6: Absolute Convergence and the Ratio and Root Tests

■ **Section 11.6** textbook exercises, *page 782*: #1, 2, 3, 4, 5, 7, 8, 9, 11, 14, 25, 26, 27, 31, 33, 35.

Definition of Absolute Convergence. A series $\sum a_n$ is called ***absolutely convergent*** if the series of absolute values $\sum |a_n|$ is convergent.

Example 1: Test for absolute convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ $\longrightarrow$ $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|(-1)^{n-1}|}{|n^2|} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p=2$ $\Rightarrow$ it converges

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ $\longrightarrow$ $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p=1$

$\Rightarrow$ It diverges.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent

Definition of Conditional Convergence. A series $\sum a_n$ is called ***conditionally convergent*** if

- it is <u>**not**</u> absolutely convergent, but
- it is convergent.

For example, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent because

1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent.

2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Example 2: Determine whether the series is absolutely convergent or conditionally convergent.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}+1}$$

<u>Absolute convergence</u>: $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1} \backsim \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

$\Downarrow$

not absolutely convergent

By LCT, also divergent

P-series with $p=\frac{1}{2}$ $\Rightarrow$ divergent

<u>Convergence</u>:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}+1}$$

$$b_n = \frac{1}{\sqrt{n}+1} \Rightarrow \lim_{n\to\infty}\frac{1}{\sqrt{n}+1} = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$$

$$\sqrt{n+1} > \sqrt{n} \Rightarrow \sqrt{n+1}+1 > \sqrt{n}+1 \Rightarrow \frac{1}{\sqrt{n+1}+1} < \frac{1}{\sqrt{n}+1}$$

$$\Rightarrow b_{n+1} < b_n$$

$\Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}+1}$ is convergent (AST)

$\Rightarrow$ Given series is conditionally convergent

THEOREM. If a series $\sum a_n$ is absolutely convergent then $\sum a_n$ is convergent.

For example, we know the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is absolutely convergent from example 1. Therefore, by the theorem above, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is automatically <u>convergent</u> without using Alternating Series Tests.

Example 3: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent or divergent.

$\hookrightarrow$ not a series with +ve terms $\Rightarrow$ cannot use CT directly.

Look at series with absolute values.

$$\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

$$0 \le |\cos n| \le 1 \Rightarrow 0 \le \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p=2$

$\Rightarrow$ It is convergent

By CT, $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is also convergent

$\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and hence convergent.

The following test is very useful in determining whether a given series is absolutely convergent

THE RATIO TEST. Given $\sum a_n$. First, we compute $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

(i) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =$ a number < 1, then the series $\sum a_n$ is absolutely convergent (and therefore $\sum a_n$ is convergent).

(ii) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ (or $= \infty$), then the series $\sum a_n$ is divergent.

(iii) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive; that is no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Example 4: Use the Ratio Test to determine whether the series is convergent or divergent

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{(n+1)^3\, 5^n}{3^n}$

$a_n = (-1)^n \dfrac{(n+1)^3\, 5^n}{3^n}$, $a_{n+1} = (-1)^{n+1} \dfrac{(n+2)^3\, 5^{n+1}}{3^{n+1}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|a_{n+1}|}{|a_n|} = \frac{(n+2)^3\, \cancel{5^{n+1}}^{\,5}}{\cancel{3^{n+1}}_{\,3}} \cdot \frac{\cancel{3^n}}{(n+1)^3\, \cancel{5^n}} = \frac{5\,(n+2)^3}{3\,(n+1)^3}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5\,(n+2)^3}{3\,(n+1)^3} = \lim_{n\to\infty} \frac{5\,\cancel{n^3}}{3\,\cancel{n^3}} = \frac{5}{3}$$

$$\frac{5}{3} > 1$$

$\Rightarrow$ By the ratio test, given series is divergent

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{n!}$

$$a_n = \frac{2^{n-1}}{n!} \Rightarrow a_{n+1} = \frac{2^{n+1-1}}{(n+1)!}$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{2^n}{(n+1)!} \cdot \frac{n!}{2^{n-1}} = \frac{2^{n-(n-1)}\, n!}{(n+1)!} = \frac{2\, n!}{(n+1)!}$$

$$= \frac{2\,\cancel{n!}}{(n+1)\,\cancel{n!}} = \frac{2}{n+1}$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1 \Rightarrow$ By ratio test, given series is abs. Conv.

The following test is convenient to apply when nth powers occur.

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}$

The Root Test. Given $\sum a_n$.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} =$ a number < 1, then the series $\sum a_n$ is absolutely convergent (and therefore $\sum a_n$ is convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ (or $= \infty$), then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

Example 5: Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

$$|a_n|^{1/n} = \left[\left(\frac{2n+3}{3n+2}\right)^n\right]^{1/n} = \frac{2n+3}{3n+2}$$

$$\Rightarrow \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \lim_{n\to\infty} \frac{2\cancel{n}}{3\cancel{n}} = \frac{2}{3} < 1$$

$\Rightarrow$ By, the root test given series is convergent