M16600 Lecture Notes

Section 11.8: Power Series

■ **Section 11.8** textbook exercises: # 3, 4, 6, 7, 9, 11, 12, 15 (these will take some time to do).

Definition of Power Series. The ***power series centered at a*** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where x is the variable and c_n's are constants called the **coefficients** of the series. Here, a is a fixed number called the **center**.

Example 1: Here are some examples of power series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \frac{(x-3)}{1} + \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} + \cdots\cdots$

The center $a = 3$. The coefficients $c_n = \frac{1}{n}$

(b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots\cdots$

The center $a = 0$. The coefficients $c_n = (-1)^n$

Note: A power series is a **function** in the variable x, where the domain is the set of all values of x such that the series converges. The outputs are series.

For example, let $f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$, i.e., $f(x) = 1 + x + x^2 + x^3 + \cdots\cdots$

$f(3) = 1 + 3 + 3^2 + 3^3 + 3^4 + \cdots\cdots$ $\boxed{f(3) \text{ is undefined.}}$

geometric series with common ratio 3 $\Rightarrow$ diverges

$f(\frac{1}{2}) = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\cdots$

geometric series with $r = \frac{1}{2} < 1 \Rightarrow$ converges to $\dfrac{1}{1-\frac{1}{2}} = 2$

$\boxed{f(\tfrac{1}{2}) = 2}$

Example 2: For what values of x is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

$$a_n = \frac{(x-3)^n}{n} \Rightarrow a_{n+1} = \frac{(x-3)^{n+1}}{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} = (x-3)^{n+1-n} \frac{n}{n+1} = (x-3)\frac{n}{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = |x-3| \frac{n}{n+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x-3| \frac{n}{n+1} = |x-3| \underbrace{\lim_{n\to\infty} \frac{n}{n+1}}_{= \lim_{n\to\infty} \frac{n}{n} = 1} = |x-3|$$

By ratio test, if $|x-3|<1$ then given series converges.

$$\Rightarrow |x-3|<1 \Rightarrow -1 < x-3 < 1 \Rightarrow 3-1 < x < 3+1$$
$$\Rightarrow 2 < x < 4$$

Therefore, if $2<x<4$, the given series is convergent.

Also, by ratio test, when $|x-3|>1$, the series is divergent.

What about $x=2$ and $x=4$? $\Rightarrow |2-3| = |4-3| = 1$

We have $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$. For $x=2$, $\displaystyle\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

For $x=4$, $\displaystyle\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

P-series with $p=1 \Rightarrow$ diverges

Alternating series with $b_n = \frac{1}{n}$

$\Rightarrow$ Converges by AST.

Thus, the series converges when $2 \le x < 4$, that is $x \in [2,4)$

x belongs to

or x lies in

The point of focus for this section is to determine for what values of x a power series is convergent. Hence, we have the following concepts.

Radius of Convergence and Interval of Convergence.

In example 2, we get $|x-3|<1$. Geometrically, this implies the ***distance*** between x and the center 3 is less than 1.

we say radius of convergence is 1

and the interval of convergence is $[2,4)$

> The ***Radius of Convergence*** of a power series is the greatest distance between x and the center a such that the series is convergent.
>
> If R is the radius of convergence, then the series is convergent for all x such that $|x-a|<R$, where a is the center of the power series.

In example 2, we find the interval $2 \leq x < 4$ for which the series is convergent. The interval $[2,4)$ is called the *interval of convergence.*

Note: To find the interval of convergence, we had to test the endpoints $x=2$ and $x=4$ separately to determine whether the series is convergent of divergent. This will be the case in general.

> The ***Interval of Convergence*** of a power series is the interval that consists of all values of x for which the series converges.

Example 3: Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Center = 0

$$a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}, \quad a_{n+1} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+1+1}} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\overset{-3}{\cancel{(-3)^{n+1}}}\ \overset{x}{\cancel{x^{n+1}}}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{\cancel{(-3)^n}\ \cancel{x^n}} = -3x\,\frac{\sqrt{n+1}}{\sqrt{n+2}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |-3x| \frac{\sqrt{n+1}}{\sqrt{n+2}} = 3|x| \underbrace{\lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n+2}}}_{=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}} = 1} = 3|x|$$

By ratio test, we must have

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow \text{Radius of convergence} = \frac{1}{3}$$

$$\Updownarrow$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

$$\boxed{x = -\frac{1}{3}} \qquad \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} \qquad \boxed{x = \frac{1}{3}}$$

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \sim \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

P-series with $P = \frac{1}{2} < 1$

$\Rightarrow$ divergent

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

Alternating series with

$$b_n = \frac{1}{\sqrt{n+1}}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$$

$$b_{n+1} = \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}} = b_n$$

$\Rightarrow$ converges by AST

Therefore, interval of converges is $\left(-\frac{1}{3}, \frac{1}{3}\right]$

Example 4: Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty} n!(x+1)^n = 1 + 1!(x+1) + 2!(x+1)^2 + 3!(x+1)^3 + \cdots$ Center $= -1$

$$a_n = n!(x+1)^n \quad \text{and} \quad a_{n+1} = (n+1)!(x+1)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(x+1)^{\cancel{n+1}}}{n!\cancel{(x+1)^n}} = \frac{(n+1)\cancel{n!}}{\cancel{n!}}(x+1) = (n+1)(x+1)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}(n+1)|x+1| = \underbrace{|x+1|}_{\text{some finite number}} \underbrace{\lim_{n\to\infty}(n+1)}_{=\infty} = \infty$$

unless $|x+1| = 0$

$\Rightarrow$ For $x \neq -1$, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty \Rightarrow$ series diverges when $x \neq -1$

For $x = -1$, $\sum_{n=0}^{\infty} n!(x+1)^n = 1 \Rightarrow$ series converges when $x = -1$

Radius of convergence $= 0$, Interval of Convergence $= \{-1\}$

Example 5: Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$a_n = \frac{x^n}{n!}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = x^{n+1-n} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \frac{x}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = \underbrace{|x|}_{\text{finite number}} \underbrace{\lim_{n\to\infty}\frac{1}{n+1}}_{=0} = 0 < 1$$

for every value of x

By ratio test, the given series converges for every value of x.

Thus, interval of convergence $= (-\infty, \infty)$ and $R = \infty$