■ **Section 11.9** textbook exercises, *page 797*: # 3, 4, 5, 6, 8, 13, 15.

In this section, we will learn how to represent certain types of functions as power series by manipulating geometric series or by differentiating or integrating such a series.

We will start with the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots\cdots \infty = \underbrace{\frac{1}{1-x}}_{f(x)}$$

$|x| < 1$ for convergence $\Rightarrow R = 1$

Thus, we get the first example of a function that is represented by a power series

Domain of $f(x)$ = Interval of convergence $= (-1, 1)$

By manipulating this first example, many other functions can also be represented as power series.

Example 1: Find a power series representation for the function and determine the interval of convergence

(a) $\frac{1}{1-x^2} = 1 + x^2 + (x^2)^2 + (x^2)^3 + \cdots\cdots \infty$

$= 1 + x^2 + x^4 + x^6 + \cdots\cdots \infty = \sum_{n=0}^{\infty} x^{2n}$

$\frac{1}{1-x^2} = \frac{a}{1-r}$ where $a = 1$, $r = x^2$

(geometric series)

Interval of Convergence

$|x^2| < 1 \Rightarrow x^2 < 1 \Rightarrow -1 < x < 1$

IOC $= (-1, 1)$

(b) $\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2}\underbrace{\frac{1}{1-\frac{x}{2}}}_{}$

geometric series sum
with $a=1$, $r=\frac{x}{2}$

$$\frac{1}{1-\frac{x}{2}} = 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots\cdots$$

$$\frac{1}{2-x} = \frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}} = \frac{1}{2}\left[1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots\right]$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \frac{x^4}{32} + \cdots\cdots\infty = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$$

For convergence, $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2 \Rightarrow -2 < x < 2$

$\Rightarrow IOC = (-2, 2)$

(c) $\frac{x}{1+2x}$

$= \frac{x}{1-(-2x)}$ } geometric series with $a=x$, $r=(-2x)$
$\left(\frac{a}{1-r}\right)$

$$= x + x(-2x) + x(-2x)^2 + x(-2x)^3 + \cdots\cdots$$

$$= \sum_{n=0}^{\infty} x(-2x)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^{n+1}$$

$r = -2x$

$\Rightarrow$ For convergence, $|-2x| < 1 \Rightarrow 2|x| < 1 \Rightarrow |x| < \frac{1}{2}$

$\Rightarrow -\frac{1}{2} < x < \frac{1}{2} \Rightarrow IOC = \left(-\frac{1}{2}, \frac{1}{2}\right)$

Differentiation and Integration of Power Series.

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Example 2:

(a) $\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \cdots$$

$$= \sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1)x^n$$

(b) $\int \left(\sum_{n=0}^{\infty} x^n\right) dx$

$$= C + \sum_{n=0}^{\infty} \int x^n\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

By differentiation or integration, we can find power series representation for more functions.

Example 3: Find a power series representation for the function and determine the radius of convergence.

R

(a) $\dfrac{1}{(1-x)^2}$. **Hint:** Note that $\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)$

$|x|<1 \Rightarrow R=1$

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1+x+x^2+x^3+\cdots\right)$$

both have same R

$$= 1+2x+3x^2+4x^3+\cdots$$

$$= \sum_{n=0}^{\infty}(n+1)x^n \Rightarrow \boxed{R=1}$$

(b) $\ln(1+x)$. **Hint:** Think about integration.

$$\ln(1+x) = \int \frac{1}{1+x}\,dx$$

$$\frac{1}{1+x} = \frac{1}{1-\underbrace{(-x)}_{r}} = 1+(-x)+(-x)^2+(-x)^3+\cdots$$

$R=1$

$|r|<1 \Rightarrow |-x|<1 \Rightarrow |x|<1$

$$= \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\int \frac{1}{1+x}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \ln(1+x) = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Find C: Put $x=0$ $\Rightarrow \ln(1) = C+0 \Rightarrow 0=C$

$$\Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

and $R=1$

(c) $\tan^{-1}(x)$. **Hint:** Think about integration.

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\, dx$$

$$\frac{1}{1+x^2} = \frac{1}{1-\underbrace{(-x^2)}_{r=-x^2}} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$r = -x^2$

$|-x^2| < 1$

$\Rightarrow x^2 < 1$

$\Rightarrow -1 < x < 1$

$\Rightarrow R = 1$

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\, dx = C + \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx$$

$$\tan^{-1}(x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Find C

Put $x=0$ on both sides.

$$\tan^{-1}(0) = C + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{2n+1}}_{=0}$$

$$\Rightarrow C = \tan^{-1}(0) = 0$$

$$\Rightarrow \tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

and $R = 1$