M16600 Lecture Notes

Section 6.1: Inverse Functions

GOALS

– Given a function $f(x)$, understanding the **inverse of** f, denote by $f^{-1}(x)$.

– Find the derivative of f^{-1} at $x = a$. *Notation*: $(f^{-1})'(a)$.

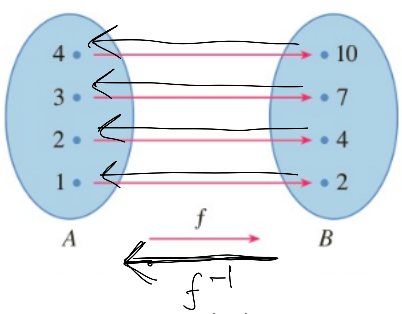
RECALL:

– The *domain* of a function is the set of all *input values*.

– The *range* of a function is the set of all *output values*.

I. Understanding Inverse Functions

Here is an example of a function $f(x)$



$f(3) = 7$

$f(1) = 2$

$f(4) = 10$

The domain of f is the set $A = \{4, 3, 2, 1\}$

The range of f is the set $B = \{10, 7, 4, 2\}$.

Definition: The ***inverse function of*** $f(x)$ is a new function, denoted by $f^{-1}(x)$.

The *domain* of f^{-1} = Range of f

The *range* of f^{-1} = Domain of f

$$f^{-1}(x) = y \iff x = f(y)$$

$$x \xrightarrow{f^{-1}} y \qquad y \xrightarrow{f} x$$

Using the example of f above, evaluate:

$f^{-1}(2) = 1$

$f^{-1}(4) = 2$

Example 1: If $f(1)=5$, $f(3)=7$, and $f(8)=3$, find $f^{-1}(7)$, $f^{-1}(5)$, $f^{-1}(3)$.

$$f^{-1}(7) = 3$$

$$f^{-1}(5) = 1$$

$$f^{-1}(3) = 8$$

Example 2: (a) Let $f(x)=x^3$, without an explicit formula of $f^{-1}(x)$, could you spot the answer for $f^{-1}(8)$?

Let $f^{-1}(8) = x$. Then by definition of inverse, $f(x) = 8$

$\Rightarrow x^3 = 8 \Rightarrow x = 2 \Rightarrow f^{-1}(8) = 2$

(b) Let $f(x)=x^3+x+1$ without an explicit formula of $f^{-1}(x)$, could you spot the answer for $f^{-1}(3)$?

Let $f^{-1}(3) = x$. Then, $f(x) = 3$

$\Rightarrow x^3 + x + 1 = 3 \Rightarrow x^3 + x = 2 \Rightarrow x = 1$

$\Rightarrow f^{-1}(3) = 1$

(c) Find the inverse function of $f(x)=x^3$.

Let $y = f^{-1}(x)$

$\Rightarrow$ By defn. of inverse function, $f(y) = x$

$\Rightarrow y^3 = x$ (find y in terms of x)

$\Rightarrow (y^3)^{1/3} = x^{1/3}$ (take cube roots on both sides) $\Rightarrow y = x^{1/3} \Rightarrow f^{-1}(x) = x^{1/3}$

Remark: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y=x$.

$x \to \boxed{f} \to \boxed{f^{-1}} \to f^{-1} \circ f(x)$ constant function 1

Cancellation Equations:

If x is in the domain of f, then $f^{-1}(f(x)) = x$, or $f^{-1} \circ f = \text{identity}$

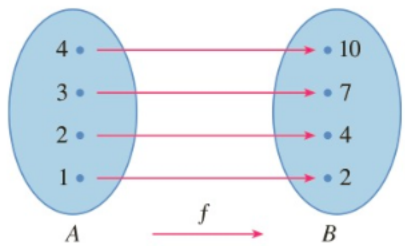
If x is in the domain of f^{-1}, then $f(f^{-1}(x)) = x$, or $f \circ f^{-1} = \text{identity}$
(range of f)

Fact: Only ***one-to-one*** functions have inverses.

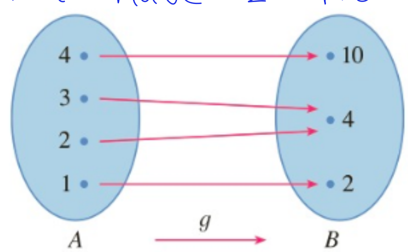
One-to-One Functions: A function f is ***one-to-one*** if

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$, or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

In other words, a function is one-to-one if every output comes from ONLY ONE input.
cannot have 2 arrows pointing to the same output.



one-one function



Not one-one

Horizontal Line Test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

II. The derivative of $f^{-1}(x)$ at $x = a$. Notation: $(f^{-1})'(a)$

Derivative Notation

Functions	Derivatives (a new function)
$f(x)$	$f'(x)$

Example 3: Let $f(x) = x^3$, then inverse function of f is $f^{-1}(x) = \sqrt[3]{x}$.

(a) Evaluate $f'(1)$

$f(x) = x^3$

$\Rightarrow f'(x) = 3x^2 \Rightarrow f'(1) = 3$

$\boxed{\frac{d}{dx}[x^n] = nx^{n-1}}$ (n any real number)

(b) Evaluate $(f^{-1})'(1)$

$f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$

$(f^{-1})'(x) = \frac{1}{3}x^{1/3 - 1} = \frac{1}{3}x^{-2/3}$

$\Rightarrow (f^{-1})'(1) = \frac{1}{3}$

There is another way of finding $(f^{-1})'(1)$ in example 3 without knowing the explicit formula of $f^{-1}(x)$.

Theorem: If f is one-to-one differentiable function with inverse function f^{-1} and $f'\big(f^{-1}(a)\big) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'\big(f^{-1}(a)\big)} \quad \text{where } a \text{ is a number} \tag{1}$$

Example 4: Let $f(x) = x^5 + x^3 + x$, use the formula (1) to find $(f^{-1})'(3)$

$$(f^{-1})'(3) = \frac{1}{f'\big(f^{-1}(3)\big)}$$

$f^{-1}(3) = ??$

Let $f^{-1}(3) = x \Rightarrow f(x) = 3 \Rightarrow x^5 + x^3 + x = 3 \Rightarrow x = 1$

$\Rightarrow f^{-1}(3) = 1$

$$\Rightarrow (f^{-1})'(3) = \frac{1}{f'(1)}$$

$f'(x) = 5x^4 + 3x^2 + 1$

$f'(1) = 5 + 3 + 1 = 9$

$$\Rightarrow (f^{-1})'(3) = \frac{1}{9}$$

Example 5: Let $f(x) = 2x + \cos x$. f is a one-to-one function. Find $(f^{-1})'(1)$.

$$(f^{-1})'(1) = \frac{1}{f'\big(f^{-1}(1)\big)}$$

Let $f^{-1}(1) = x \Rightarrow f(x) = 1 \Rightarrow 2x + \cos x = 1$

At $x = 0$ we have $2(0) + \cos 0 = 1$

$\Rightarrow f^{-1}(1) = 0 \Rightarrow (f^{-1})'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$.

$f'(x) = 2 - \sin x$

$f'(0) = 2 - \sin 0 = 2$

Section 6.1 Exercises, *page 406*: # 3, 5, 7, 17, 23, 39, 40, 41, 42, 43