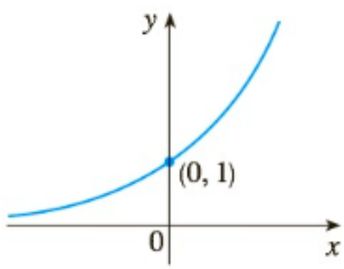
M16600 Lecture Notes

Section 6.2: Exponential Functions and Their Derivatives

SUMMARY:

- The general Exponential Functions $f(x) = b^x$ and their properties.
- The Natural Exponential Functions $f(x) = e^x$ and its calculus facts
- The derivative of e^x: $\dfrac{d}{dx}(e^x) = e^x$
- The integral (or antiderivative) of e^x: $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\lim_{x\to\infty} e^x = \infty$ and $\displaystyle\lim_{x\to-\infty} e^x = 0$
- The graph of $y = e^x$



I. Exponential Functions

Definition: An ***exponential function*** is a function of the form

$$f(x) = b^x \qquad (b \text{ is fixed})$$

where b is a positive constant. , b is real number and $b > 0$

Warning: Exponential functions are not the same as ***power functions***

Power functions : $g(x) = x^n$ (n is fixed)

Examples: $g(x) = x^2$, $g(x) = x^5$, $g(x) = x^{-1}$, ...

- If $x = n$, a positive number, then

$$b^n = \underbrace{b \cdot b \cdot b \cdots b \cdot b}_{n \text{ factors}}$$

$$b^{-n} = \frac{1}{b^n}$$

Examples of exponential functions

$f(x) = 2^x$, $f(x) = 5^x$

$f(x) = 0.5^x$, $f(x) = \left(\frac{1}{3}\right)^x$

- If $x = 0$, then $b^0 = 1$
- If x is a rational number then $b^x = b^{n/d} = \sqrt[d]{b^n}$

$\boxed{f(x) = (-2)^x}$ X base -2 is not positive

We can also define b^x for any irrational number x (see the discussion in the textbook, pages 408 and 409).

Properties of Exponential Functions: If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. If $a, b > 0$ and $x, y \in \mathbb{R}$, then we have the following

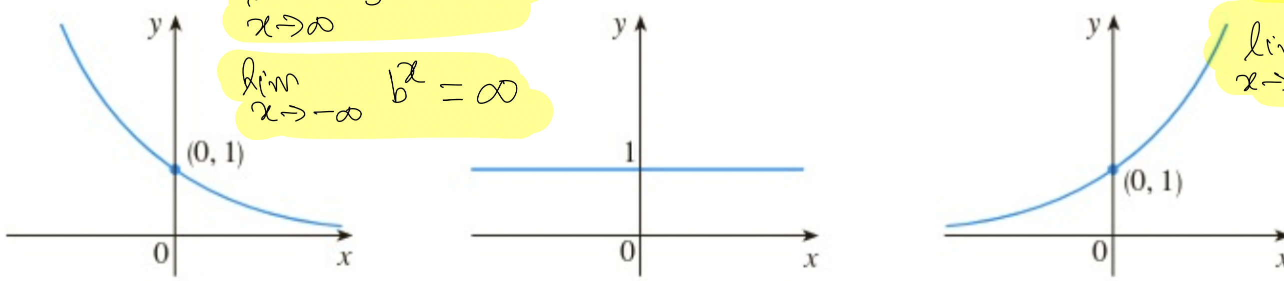
- $b^x > 0$ for all x
- **Laws of Exponents:** $b^{x+y} = b^x b^y$, $\quad b^{x-y} = \dfrac{b^x}{b^y}$, $\quad (b^x)^y = b^{xy}$, $\quad (ab)^x = a^x b^x$.
- Graphs and limits

$\lim_{x\to\infty} b^x = 0$

$\lim_{x\to-\infty} b^x = \infty$

$\lim_{x\to\infty} b^x = \infty$

$\lim_{x\to-\infty} b^x = 0$



(a) $y = b^x,\ 0 < b < 1$ (b) $y = 1^x$ (c) $y = b^x,\ b > 1$

(a) decreasing and convex function

$f'(x) < 0$

$f''(x) > 0$

(c) increasing and convex func.

$f'(x) > 0$

$f''(x) > 0$

Example 1: (a) Find $\lim_{x\to\infty} (2^{-x} - 1)$.

$$\lim_{x\to\infty} 2^{-x} - \lim_{x\to\infty} 1$$

$$= \lim_{x\to\infty} \frac{1}{2^x} - 1$$

$$= \frac{1}{\lim_{x\to\infty} 2^x} - 1$$

$$= \frac{1}{\infty} - 1 = 0 - 1 = -1$$

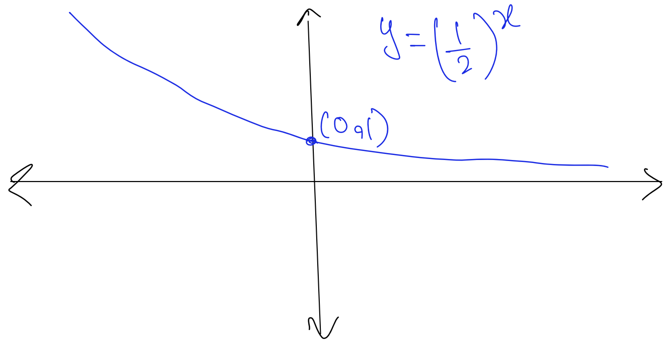
Alternatively

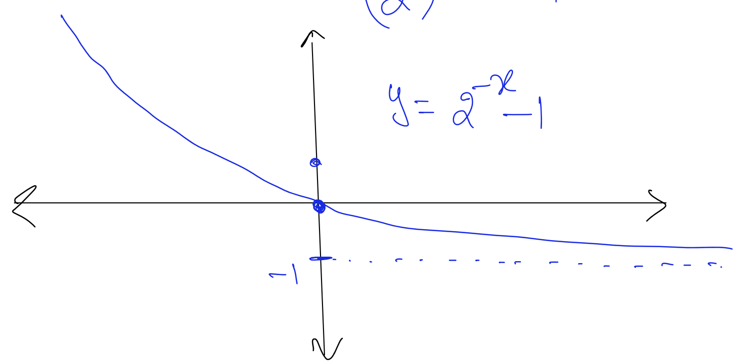
$$= \lim_{x\to\infty} \left(\frac{1}{2}\right)^x - 1$$

$$= 0 - 1$$

$$= -1$$

(b) Sketch the graph of the function $y = 2^{-x} - 1$. $\Rightarrow y = \left(\frac{1}{2}\right)^x - 1$





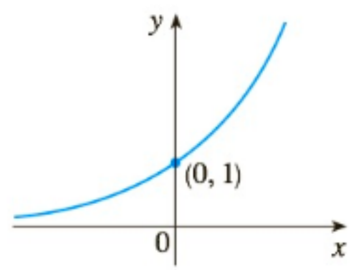
Introducing the *Natural Exponential Function* $f(x) = e^x$, where e is an irrational number. Its approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536 \simeq 2.72$$

Read the discussion on *Derivatives of Exponential Functions*, page 412, for the motivation of defining the number e.

Some *Calculus facts* of the natural exponential function e^x. $\boxed{e>1}$

- The derivative of e^x: $\dfrac{d}{dx}\big(e^x\big) = e^x$
- The integral of e^x: $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\lim_{x\to\infty} e^x = \infty$ and $\displaystyle\lim_{x\to-\infty} e^x = 0$
- The graph of $y = e^x$ *[Figure 14, section 6.2, textbook]*



Example 2: Rewrite the following expression into the form e^P, where P is some algebraic expression.

1. $e^x e^{x^2} = e^{x^2+x} = e^{x+x^2}$

2. $\dfrac{1}{e^x} = \dfrac{e^0}{e^x} = e^{0-x} = e^{-x}$

3. $\dfrac{e^{3x}}{e^2} = e^{3x-2}$

4. $(e^{x^2})^4 = e^{4x^2}$

$$\frac{d}{dx}\left[e^x\right] = e^x$$

Example 3: Differentiate

(a) $f(x) = e^{-3} + x^{-3} - e^x + e^{14}$

$$f'(x) = \left[e^{-3}\right]' + \left[x^{-3}\right]' - \left[e^x\right]' + \left[e^{14}\right]'$$

$$= 0 + (-3)\,x^{-3-1} - e^x + 0$$

$$= -3x^{-4} - e^x$$

(b) $g(x) = e^{x^7-4x}$ $= e^z$ where $z = x^7 - 4x$

$$g'(x) = \frac{d}{dx}\left(e^z\right) = \frac{d}{dz}\left(e^z\right)\frac{dz}{dx}$$

$$= e^z \cdot \frac{d}{dx}\left(x^7 - 4x\right) = e^z\left(7x^6 - 4\right)$$

$$= e^{x^7-4x}\left(7x^6 - 4\right)$$

(c) $y = \sqrt{x}\,e^{x/5} - \sin(5x)$

$$y' = \left[\sqrt{x}\,e^{x/5}\right]' - \left[\sin(5x)\right]'$$

$$= \left(\sqrt{x}\right)' e^{x/5} + \sqrt{x}\left[e^{x/5}\right]' - \left[\sin(5x)\right]'$$

$z = 5x$

$\cos z \cdot \frac{dz}{dx}$

$$= \frac{1}{2\sqrt{x}}\,e^{x/5} + \sqrt{x}\;e^{x/5}\cdot\frac{1}{5} - \cos(5x)\cdot 5$$

$$= e^{x/5}\left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{5}\right) - 5\cos(5x)$$

(d) $h(x) = \dfrac{(e^x)^{23}}{1 - e^x}$ $= e^{23x}/(1-e^x)$

$$h'(x) = \frac{\left[e^{23x}\right]'(1-e^x) - \left(e^{23x}\right)\left[1-e^x\right]'}{(1-e^x)^2} = \frac{23\,e^{23x}(1-e^x) - e^{23x}(-e^x)}{(1-e^x)^2}$$

$$= \frac{23e^{23x} - 23e^{24x} + e^{24x}}{(1-e^x)^2} = \frac{23e^{23x} - 22e^{24x}}{(1-e^x)^2}$$

(e) $f(t) = \tan(e^t)$

$$f'(t) = \frac{d}{dt}\Big[\mathrm{Tan}(\underbrace{e^t}_{z})\Big] = \frac{d}{dt}\left[\mathrm{Tan}(z)\right]$$

$$= \frac{d}{dz}(\mathrm{Tan}\,z)\cdot\frac{dz}{dt} = \sec^2 z \cdot \frac{d}{dt}(e^t)$$

$$= e^t \sec^2(e^t) \quad [\text{Replace } z = e^t]$$

(f) $y = e^{4\sin(x)}$

Let $z = 4\sin x$

$$\frac{dy}{dx} = \frac{d}{dx}\left[e^z\right] = \frac{d}{dz}\left[e^z\right]\cdot\frac{dz}{dx} = e^z \cdot \frac{d}{dx}(4\sin x)$$

$$= e^{4\sin x}\cdot(4\cos x)$$

Alternatively

$$\frac{d}{dx}\left(e^{4\sin x}\right) = e^{4\sin x}\cdot(4\sin x)' = 4\cos x\, e^{4\sin x}$$

Example 4: Evaluate the integral

(a) $\int (e^x - x^e + 1)\,dx$

$$= \int e^x\,dx - \int x^e\,dx + \int 1\,dx$$

$$= e^x - \frac{x^{e+1}}{e+1} + x + C$$

$$\boxed{\int e^x\,dx = e^x + C}$$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)}$$

(b) $\int_0^1 \frac{3}{e^x}\,dx$

$$= \int_0^1 3\, e^{-x}\, dx$$

Let $z = -x$

$$\Rightarrow \frac{dz}{dx} = -1 \Rightarrow dz = -dx$$

$$\Rightarrow -dz = dx$$

$$= 3\int_0^1 e^{-x}\, dx$$

$$= 3\int_{z=-0}^{z=-1} e^{z}\,(-dz) = -3\int_0^{-1} e^{z}\, dz = -3\, e^{z}\Big|_0^{-1}$$

$$= -3e^{-1} - (-3e^{0})$$

$$= -3e^{-1} + 3 = 3(1-e^{-1})$$

(c) $\int x e^{x^2}\,dx$

Let $z = x^2 \Rightarrow \frac{dz}{dx} = 2x \Rightarrow dz = 2x\,dx \Rightarrow \boxed{\frac{1}{2}dz = x\,dx}$

$$\int x e^{x^2}\, dx = \int e^{x^2}\cdot \underbrace{x\, dx}$$

$$= \int e^{z}\cdot \frac{1}{2}dz = \frac{1}{2}\int e^{z}\, dz = \frac{1}{2}e^{z} + C$$

Substitute $z = x^2$

$$= \frac{1}{2}e^{x^2} + C$$

(d) $\int e^x \sqrt[4]{e+e^x}\,dx$

Substitute $u = e + e^x \Rightarrow \frac{du}{dx} = e^x \Rightarrow du = \underbrace{e^x\, dx}$

$$I = \int e^x \sqrt[4]{e+e^x}\, dx = \int \sqrt[4]{\underbrace{e+e^x}_{u}}\;\underbrace{e^x\, dx}_{du}$$

$$= \int \sqrt[4]{u}\, du = \int u^{1/4}\, du = \frac{u^{\frac{1}{4}+1}}{\frac{1}{4}+1} + C = \frac{u^{5/4}}{5/4} + C$$

$$= \frac{4}{5} u^{5/4} + C$$

$$= \frac{4}{5} (e + e^x)^{5/4} + C$$

(e) $\displaystyle\int_{\pi/2}^{\pi} \sin x \, e^{\cos x} \, dx$

Substitute $u = \cos x$

$$\Rightarrow \frac{du}{dx} = -\sin x \Rightarrow du = -\sin x \, dx \Rightarrow -du = \sin x \, dx$$

$$I = \int_{\pi/2}^{\pi} \sin x \, e^{\cos x} \, dx = \int_{\pi/2}^{\pi} e^{\overset{u}{\boxed{\cos x}}} \underbrace{\sin x \, dx}_{-du}$$

$x = \frac{\pi}{2} \Rightarrow u = \cos \frac{\pi}{2} = 0$ and $x = \pi \Rightarrow u = \cos \pi = -1$

$$\Rightarrow I = \int_0^{-1} e^u (-du) = -\int_0^{-1} e^u \, du = \int_{-1}^{0} e^u \, du$$

$$= e^u \Big|_{-1}^{0} = e^0 - e^{-1} = 1 - \frac{1}{e}$$

(f) $\displaystyle\int \frac{2e^x}{(3 + e^x)^3} \, dx$

Substitute $u = 3 + e^x \Rightarrow \dfrac{du}{dx} = e^x \Rightarrow \underbrace{du = e^x \, dx}$

$$\int \frac{2e^x}{(3+e^x)^3} dx = \int 2 (\underbrace{3+e^x}_{u})^{-3} \underbrace{e^x \, dx}_{du}$$

$$= \int 2 u^{-3} \, du = 2 \int u^{-3} \, du = 2 \frac{u^{-3+1}}{-3+1} + C$$

$$= \cancel{2} \frac{u^{-2}}{-\cancel{2}} + C = -\frac{1}{u^2} + C = \frac{-1}{(3+e^x)^2} + C$$

Section 6.2 exercises, *page 418*, #7, <u>9</u>, 23, 24, 26, 31, 33, 37, 39, 42, 83, 85, 86, 87, 90, 91, 94. If computing the derivative, you don't need to simplify the answers. Underline problems are optional.