M16600 Lecture Notes

Sections 6.3: Logarithmic Functions

OBJECTIVES

- Switch back and forth between exponential equations and logarithmic equations
- Know the notation for the natural logarithmic function and how to use it
- Memorize and utilize the properties of logarithm
- Know the graph of logarithmic functions and their limit equations
- Know the change of base formula

The *exponential function* $f(x) = b^x$ has an inverse function $f^{-1}(x)$, which is called ***logarithmic function with base b***. Recalling the relationship

$$f^{-1}(x) = y.$$

Then we have $\Rightarrow x = f(y)$. This means $\underbrace{y = \log_b x}_{f^{-1}(x)} \iff \underbrace{x = b^y}_{f(y)}$

The ***natural logarithm*** is the logarithm with base e and has a special notation: $\boxed{\log_e = \ln x}$. Then

$$\boxed{\ln x = y \iff e^y = x}$$

Logarithm base 10 also has a special notation: $\boxed{\log_{10} = \log}$. , $\log_2 = lg$

Example 1: Change the exponential equations to logarithmic equations

(a) $5^x = 35$

$\Rightarrow x = \log_5 35$

(b) $\frac{1}{2} = e^{-0.016t}$

$-0.016t = \log_e \frac{1}{2}$

Example 2: Change the logarithmic equations to exponential equations

(a) $\log_3 81 = 4$

$3^4 = 81$

(b) $\ln(2x - 1) = 3$

$e^3 = 2x - 1$

Example 3: Evaluate (a) $\log_3 9$ (b) $\log_{25} 5$.

(a) $\log_3 9 = 2$

(b) $\log_{25} 5 = \frac{1}{2}$

• **Properties of Logarithmic Functions**: If $b > 1$, the function $f(x) = \log_b x$ is one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If x, $y > 0$ and p is any real number, then we have the ***Laws of Logarithms*** as follows:

(i) $\log_b(xy) = \log_b x + \log_b y$

(ii) $\log_b \frac{x}{y} = \log_b x - \log_b y$

(iii) $\log_b x^p = p \log_b x$

Cancellation Equations:

$$\log_b(b^x) = x \quad \text{for } x \in \mathbb{R}$$
$$b^{log_b x} = x \quad \text{for every } x > 0$$

Example 4: Expand $\ln \sqrt{\dfrac{x+1}{x^2 y}}$

$$\ln \sqrt{\frac{x+1}{x^2 y}} = \ln \left(\frac{x+1}{x^2 y}\right)^{1/2}$$

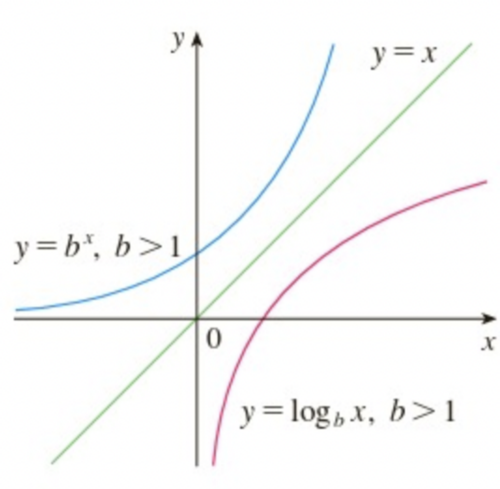
$$= \frac{1}{2} \ln \left(\frac{x+1}{x^2 y}\right) \qquad \text{[By Property (iii)]}$$

$$= \frac{1}{2}\left[\ln(x+1) - \ln(x^2 y)\right] = \frac{1}{2}\left[\ln(x+1) - \left(\ln(x^2) + \ln(y)\right)\right]$$

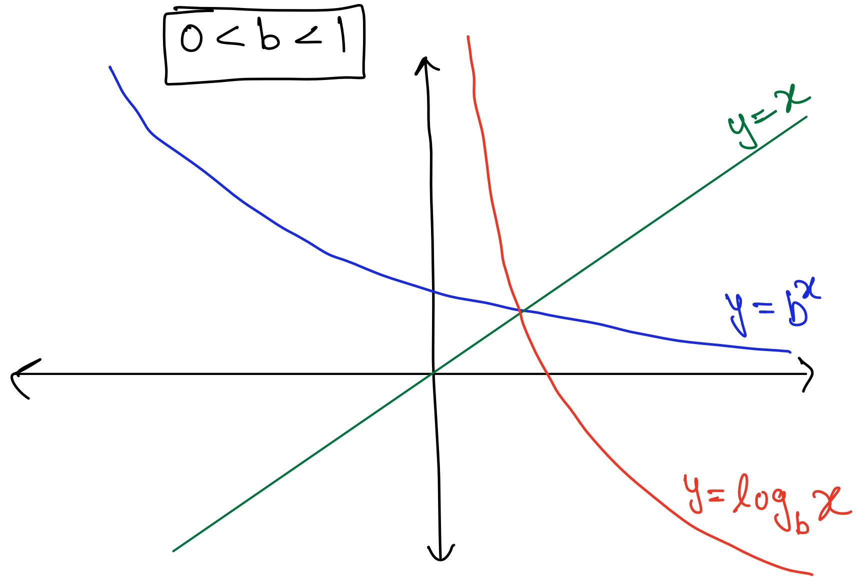
$$= \frac{1}{2}\left[\ln(x+1) - \ln(x^2) - \ln(y)\right] = \frac{1}{2}\left[\ln(x+1) - 2\ln(x) - \ln y\right]$$

$$= \frac{1}{2}\ln(x+1) - \ln(x) - \frac{1}{2}\ln(y)$$

• **The Graph of the Logarithm Function and the Exponential Function on the Same xy-plane, $b > 1$**



$\boxed{0 < b < 1}$

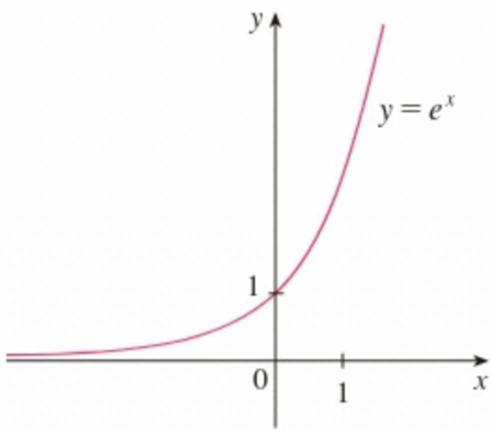


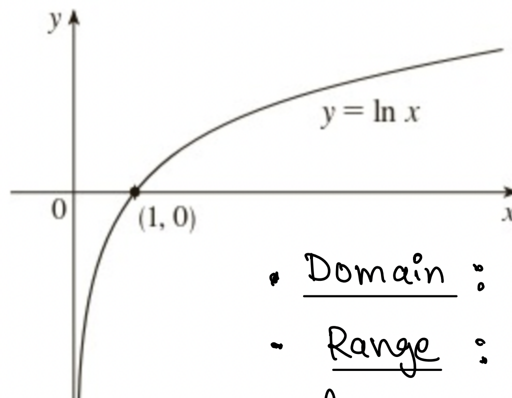
Next, we focus on **the natural logarithm** $\ln(x)$, which is log base e. All of what we know about the general logarithmic function are applied to $\ln x$.

$\ln 1 = 0$ because $e^0 = 1$ $\quad (\log_b 1 = 0)$ for any base b

$\ln e = 1$ because $e^1 = e$

$$\left.\begin{aligned} \ln(e^x) &= x \\ e^{\ln x} &= x \end{aligned}\right\} \text{Cancellation equations}$$





- $\underline{\text{Domain}}$: $(0, \infty)$
- $\underline{\text{Range}}$: $\mathbb{R}$
- $\lim_{x \to \infty} \ln(x) = \infty$
- $\lim_{x \to 0^+} \ln(x) = -\infty$
- $\ln(x)$ is an increasing function
- $\ln(x)$ is concave downwards

Example 5: Find x if $\ln x = 5$

$\Rightarrow x = e^5$

Example 6: Solve the equation $e^{5-3x} + 4 = 14$

$\Rightarrow e^{5-3x} = 14 - 4$

$\Rightarrow e^{5-3x} = 10$

$\Rightarrow \ln e^{5-3x} = \ln(10)$

Cancellation equation

$\Rightarrow 5 - 3x = \ln 10$

$\Rightarrow -3x = \ln 10 - 5 \Rightarrow \dfrac{-3x}{-3} = \dfrac{\ln 10 - 5}{-3}$

$\Rightarrow x = \dfrac{-1}{3}\ln 10 + \dfrac{5}{3}$

Example 7: Express $\ln a + \frac{1}{5}\ln b - \ln(a+b)$ as a single logarithm.

$$\ln a + \frac{1}{5}\ln b - \ln(a+b) = \ln a + \ln b^{1/5} - \ln(a+b)$$
$$= \ln(a b^{1/5}) - \ln(a+b)$$
$$= \ln\left(\frac{a b^{1/5}}{a+b}\right)$$

Change of Base Formula: For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b} = \frac{\log_c x}{\log_c b} \quad \text{where } c > 0 \ (c \neq 1)$$

Example 8: Evaluate $\log_8 5$

$$\log_8 5 = \frac{\ln 5}{\ln 8} = \frac{\ln 5}{\ln 2^3}$$
$$= \frac{\ln 5}{3\ln 2} = \frac{0.477}{3(0.3)}$$
$$= \frac{0.477}{0.9} = 0.53$$

$$\frac{0.477}{0.900} = \frac{477}{900}$$
$$= \frac{53}{100} = 0.53$$

Section 6.3 exercises, *page 426*, #3, 4, 5, 9, 11, 13, <u>15</u>, 17, <u>19</u>, 27, 29, 31, <u>47</u>, <u>51</u>.