M16600 Lecture Notes

Sections 6.4: Derivatives of Logarithmic Functions

SUMMARY

New Differentiation Formulas

- $\dfrac{d}{dx}\left(\ln x\right) = \dfrac{1}{x}$
- $\dfrac{d}{dx}\left(\log_b x\right) = \dfrac{1}{x\,\ln b}$
- $\dfrac{d}{dx}\left(b^x\right) = b^x\,\ln b$

New Integral Formulas

- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b} + C$, where $b \neq 1$.

New Differentiation Technique: Logarithmic Differentiation

- **The Derivative and Integral Formula of** $\ln x$

$$\boxed{\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}} \qquad \boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

Example 1: Differentiate

(a) $f(x) = \sqrt{\ln x}$

Let $z = \ln x \Rightarrow f(x) = \sqrt{z}$

$$f'(x) = \frac{d}{dx}\left(\sqrt{z}\right) = \frac{d}{dz}\left(\sqrt{z}\right)\frac{dz}{dx} = \frac{1}{2\sqrt{z}} \cdot \frac{d}{dx}(\ln x)$$

$$= \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

(b) $g(x) = \ln(\sin x)$

Let $z = \sin x \Rightarrow g(x) = \ln z$

$$g'(x) = \frac{d}{dx}\left(\ln z\right) = \frac{d}{dz}(\ln z)\,\frac{dz}{dx}$$

$$= \frac{1}{z} \cdot \frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \frac{\cos x}{\sin x}$$

$$= \cot x$$

Example 2: Evaluate

$$\cancel{dx} \cdot \frac{du}{\cancel{dx}} = 2x \cdot dx$$

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx$

Let $u = x^2+1 \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2} du = \frac{1}{\cancel{2}} \cdot \cancel{2}x\,dx$

$\Rightarrow \underbrace{\frac{1}{2} du = x\,dx}$

$$I = \int \frac{x}{x^2+1}\,dx = \int \frac{1}{x^2+1} \cdot x\,dx$$

$$= \int \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C$$

$$= \frac{1}{2} \ln|x^2+1| + C$$

(b) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx = \int_1^e \underbrace{\ln x}_{u} \cdot \underbrace{\frac{1}{x} \cdot dx}_{=du}$

Let $u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow \underbrace{du = \frac{1}{x} dx}$

$x=1 \Rightarrow u = \ln 1 = 0$

$x=e \Rightarrow u = \ln e = 1$

$$\Rightarrow I = \int_{u=0}^{u=1} u \cdot du$$

$$= \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}$$

- **Logarithmic Differentiation**

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. This method is called ***Logarithmic Differentiation***

Example 3: Use **Logarithmic Differentiation** to find the derivative of

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

Step 1 Take ln on both sides and simplify using log Properties

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5} \Rightarrow \ln y = \ln\left[\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right]$$

$$\Rightarrow \ln y = \ln\left[x^{3/4}\sqrt{x^2+1}\right] - \ln\left[(3x+2)^5\right]$$

$$= \ln x^{3/4} + \ln(x^2+1)^{1/2} - \ln(3x+2)^5$$

$$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Step 2 Diff. both sides w.r.t. x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\left(\frac{1}{x}\right) + \frac{1}{\cancel{2}}\left(\frac{\cancel{2}x}{x^2+1}\right) - 5\left(\frac{3}{3x+2}\right)$$

$$= \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$$

Step 3 Multiply both sides by y to get $\frac{dy}{dx}$

$$\cancel{y}\cdot\frac{1}{\cancel{y}}\frac{dy}{dx} = y\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

Example 4: Differentiate

(a) $y = x^2$

Step 1 $\Rightarrow \ln y = \ln x^2 \Rightarrow \ln y = 2\ln x$

Step 2 $\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$

Step 3 $\Rightarrow \frac{dy}{dx} = y \cdot \frac{2}{x} = x^2 \cdot \frac{2}{x} = 2x$

(b) $y = e^x$

Step 1 $\Rightarrow \ln y = \ln e^x \Rightarrow \ln y = x \ln e \Rightarrow \ln y = x$

Step 2 $\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1$ Step 3 $\Rightarrow \frac{dy}{dx} = y \Rightarrow \frac{dy}{dx} = e^x$

(c) $y = x^{\sqrt{x}}$

Step 1 $\Rightarrow \ln y = \ln x^{\sqrt{x}} \Rightarrow \ln y = \sqrt{x}\ln x$

Step 2 $\Rightarrow \frac{1}{y}\frac{dy}{dx} = [\sqrt{x}\ln x]'$

$$= [\sqrt{x}]'\ln x + \sqrt{x}[\ln x]'$$

$$= \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} = \frac{\sqrt{x}\ln x + 2\sqrt{x}}{2x}$$

$$\frac{\sqrt{x}\ln x}{2\sqrt{x}\cdot\sqrt{x}} + \frac{2\sqrt{x}}{2x} = \frac{\sqrt{x}\ln x}{2x} + \frac{2\sqrt{x}}{2x}$$

Step 3 $\Rightarrow \frac{dy}{dx} = y\left(\frac{\sqrt{x}\ln x + 2\sqrt{x}}{2x}\right)$

$$\Rightarrow \frac{dy}{dx} = x^{\sqrt{x}}\left(\frac{\sqrt{x}\ln x + 2\sqrt{x}}{2x}\right)$$

$$= x^{\sqrt{x}}\,\frac{\cancel{\sqrt{x}}(\ln x + 2)}{2\cancel{x}_{\sqrt{x}}} = \frac{x^{\sqrt{x}}(\ln x + 2)}{2\sqrt{x}}$$

Example 5: Evaluate

(a) $\dfrac{d}{dx}\big(\log_2 x\big)$

$$= \frac{1}{x \ln 2}$$

(b) $\dfrac{d}{dx}(2^{2x})$

$$\circledast \; 2^{2x} = (2^2)^x = 4^x$$

$$= \frac{d}{dx}(4^x)$$

$$= 4^x \ln 4 = 2^{2x} \ln 4$$

$$\circledast \ln 4 = \ln 2^2 = 2 \ln 2$$

$$= 2^{2x}(2 \ln 2)$$

$$= 2^{2x+1} \ln 2$$

(c) $\displaystyle\int 2^x \, dx$

$$= \frac{2^x}{\ln 2} + C$$

Section 6.4 exercises, *page 436*, #3, <u>5</u>, 7, 9, 10, 11, 15, <u>13</u>, <u>35</u>, 43, 45, <u>47</u>, 49, 51, <u>53</u>, 71, 73, 75, 77, 78, 80. Hint on #13: rewrite $G(y)$ first.