■ **Section 6.5** exercises, *page 471*, #1, 3, 5(a)(b)(c), 8(a)(b)(c)(d), 9

SUMMARY

- Solve the Population Growth problems
- Solve the Radioactive Decay problems.

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individual in population of animals or bacteria at time t, then we can expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$, i.e., $f'(t) = kf(t)$ for some constant k.

In general, if
$y(t)$ is *the value of a quantity* y *at time* t and
the rate of change *of* y *with respect to* t is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky, \qquad \text{where } k \text{ is a constant.} \tag{1}$$

This equation is called a ***differential equation*** because it involves an unknown function y and its derivative $\frac{dy}{dt}$.

Solving a differential equation means finding the solution, or the original function $y(t)$, such that equation 1 is satisfied.

Theorem: The only solutions of the differential equation 1 are the exponential functions

$$y(t) = Ce^{kt}, \qquad \text{where } C = y(0), \text{ the } \textit{initial value} \text{ of the function } y.$$

Or we can simply write: $y(t) = y(0)e^{kt}$.

$$\frac{dy}{dt} = \frac{d}{dt}\left(Ce^{kt}\right) = C\frac{d}{dt}\left(e^{kt}\right)$$

$$= Ce^{kt} \cdot \frac{d}{dt}(kt)$$

$$= Cke^{kt} = k\left(Ce^{kt}\right) = ky$$

◇ **Population Growth**: In the context of population growth, where $P(t)$ is the size of the population at time t, we can write

$$\frac{dP}{dt} = kP \quad \text{or} \quad k = \frac{1}{P}\frac{dP}{dt} \quad \text{which is the growth rate divided by the population size.}$$

Therefore, the constant k is called ***relative growth rate***.

The expression of the population function is

$$\boxed{P(t) = P(0)e^{kt}, \quad \text{where } P(0) \text{ is the initial population.}}$$

Example 1: The common inhabitant of human intestines is the bacterium *Escherichia coli*, named after the German pediatrician Theodor Escherich, who identified it in 1885. A cell of this bacterium in a nutrient-broth medium divideds into two cells every 20 minutes. The initial population of a culture is 50 cells.

(a) Find the relative growth rate.

(b) Find an expression for the number of cells after t hours.

(c) Find the number of cells after 6 hours.

(d) Find the rate of growth after 6 hours.

(e) When will the population reach a ~~minion~~ million cells?

$P(t) = P(0)e^{kt}$. $\quad P(0) = 50$

$= 50e^{kt}$

(a) $P(20) = 2(50) = 100$

$\quad \| $

$50e^{k(20)} = 100 \Rightarrow 50e^{20k} = 100 \Rightarrow e^{20k} = \frac{100}{50} = 2$

$\Rightarrow \ln(e^{20k}) = \ln(2) \Rightarrow 20k = \ln 2 \Rightarrow k = \frac{\ln 2}{20 \text{ mins.}}$

$\Rightarrow k = \frac{\ln 2}{\frac{20}{60} \text{hrs}} = \frac{\ln 2}{1/3}$

$\Rightarrow k = 3\ln 2$ per hour.

(b) $P(t) = 50\, e^{kt} = 50\, e^{(3\ln 2)t}$

$= 50\, e^{3t \ln 2} = 50\, e^{\ln (2)^{3t}} = 50\,(2)^{3t}$

$$P(t) = 50\,(2)^{3t} = 50\,(8)^t$$

(c) $P(6) = 50\,(8)^6$

(d) $\frac{dP}{dt} = P'(t)$ (rate of growth)

$P'(6) = ?$

$P'(t) = \frac{d}{dt}\left(50\,(8)^t\right)$

$= 50\,\frac{d}{dt}\left(8^t\right) = 50\left(8^t \ln 8\right)$

$\Rightarrow P'(6) = 50\,(\ln 8)\, 8^6$ cells/hour

Alternatively $P'(t) = k\,P(t)$

$\Rightarrow P'(6) = k\,P(6) = 3\ln 2\left(50\,(8)^6\right)$ cells/hour.

(e) Find t for which $P(t) = 10^6$

$\Rightarrow 50\,(8)^t = 10^6 \Rightarrow 8^t = \frac{10^6}{50} = 10^4 \times \frac{100}{50}$

$\Rightarrow 8^t = 20000 \Rightarrow \ln 8^t = \ln(20\,000) \Rightarrow t \ln 8 = \ln 20000$

$\Rightarrow t = \frac{\ln 20000}{[illegible]}$ hours.

$\ln 8$

◇ **Radioactive Decay**: Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass $m(0)$ of the substance after time t, then

$$\frac{dm}{dt} = km \qquad \text{where } k \text{ is a } \textit{negative} \text{ constant.}$$

In other words, radioactive substances decay at a rate proportional to the remaining mass. This means ***the expression of the remaining mass*** m ***after time*** t is given by

$$\boxed{m(t) = m(0)e^{kt}}$$

Physicists express the rate of decay in terms of ***half-life***, the time required for half of any given quantity to decay.

Example 2: The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100mg. Find a formula for the mass of the sample that remains after t year.

(b) Find the mass after 1000 years.

(c) When will the mass be reduced to 30 mg?

t_{half} (half-life) $= 1590$ years.

$m(0) = 100 \quad \Rightarrow m(t) = 100\, e^{kt}$

(a) In 1590 years, mass becomes $\frac{1}{2}(100) = 50$ mg

$\Rightarrow m(1590) = 50 \quad \Rightarrow 100\, e^{k(1590)} = 50$

$\Rightarrow e^{k(1590)} = \frac{50}{100} \quad \Rightarrow e^{k(1590)} = \frac{1}{2}$

$\Rightarrow \ln\left(e^{1590k}\right) = \ln\left(\frac{1}{2}\right) \quad \Rightarrow 1590k \underbrace{\ln e}_{1} = \underbrace{\ln 2^{-1}}_{-\ln 2}$

$\Rightarrow 1590k = -\ln 2$

$\Rightarrow k = -\frac{\ln 2}{1590}$

$\Rightarrow m(t) = 100\, e^{-\frac{\ln 2}{1590}t}$

$-t$

$$= 100\, e^{\frac{-t}{1590}\ln 2} = 100\, e^{\ln 2^{\frac{-t}{1590}}} = 100\,(2)^{\frac{-t}{1590}}$$

(b) $m(1000) = 100\,(2)^{\frac{-1000}{1590}} = 100\,(2)^{\frac{-100}{159}}$ mg.

(c) t for which $m(t) = 30$ mg.

$$\Rightarrow 100\,(2)^{\frac{-t}{1590}} = 30$$

$$\Rightarrow (2)^{\frac{-t}{1590}} = \frac{30}{100} = \frac{3}{10}$$

$$\Rightarrow \ln (2)^{\frac{-t}{1590}} = \ln\left(\frac{3}{10}\right) = -\ln \frac{10}{3}$$

$$\Rightarrow \cancel{-}\frac{t}{1590}\ln 2 = \cancel{-}\ln\frac{10}{3}$$

$$\Rightarrow t = \frac{1590 \ln \frac{10}{3}}{\ln 2} = \frac{1590(\ln 10 - \ln 3)}{\ln 2} \text{ yrs.}$$