■ **Section 6.6** exercises, *page 481*: #1, 2, 3, 4, 5, 7, 12, 13, 22, 23, 25, 27, 31, 33, 59, 61, 65, 64, 67.

GOALS

- Compute the values of the **inverse trigonometric functions**, e.g., $\sin^{-1}(\frac{1}{2})$, $\cos^{-1}(0)$, $\tan^{-1}(\sqrt{3})$, etc.
- Compute or simplify expressions such as $\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$, $\cos\left(\tan^{-1} x\right)$, etc.
- Compute derivatives and integrals involving inverse trigonometric functions.

In this section, we explore the inverse functions of trigonometric functions. The functions $\sin(x)$, $\cos(x)$, $\tan(x)$ are not one-to-one over their domains. However, if we restrict their domains, they will be one-to-one on the restricted domain. We then can find their inverse functions.

◇ **Inverse Sine Function**. *Notation*: $\sin^{-1}(x)$ or $\arcsin(x)$

$\sin\theta$ is one-to-one for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Thus, we have

$$\boxed{\sin^{-1} x = \theta \iff \sin\theta = x \quad \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$$

$\boxed{\textbf{Note: } \sin^{-1} \neq \frac{1}{\sin x}}$ $\longrightarrow (\sin x)^{-1}$

Example 1: Evaluate (a) $\sin^{-1}(\frac{1}{2})$ (b) $\tan\left(\arcsin\frac{1}{3}\right)$

(a) For what angle θ do we have

$\sin^{-1}(\frac{1}{2}) = \theta \Rightarrow \sin\theta = \frac{1}{2} \Longrightarrow \theta = \frac{\pi}{6}$ or 30°

$\Rightarrow \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$

(b) $\tan\left(\overbrace{\arcsin\frac{1}{3}}^{\theta}\right) \Longrightarrow$ Let $\theta = \arcsin\left(\frac{1}{3}\right)$

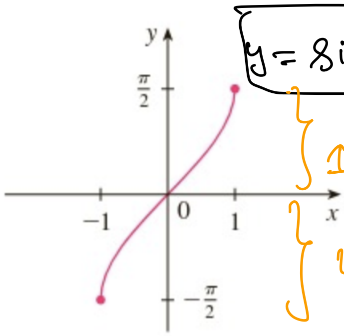
$\Rightarrow \sin\theta = \frac{1}{3}$; $\frac{1}{3} > 0$ $\Rightarrow \theta$ is in first quadrant

Since θ is in 1^{st} quadrant

$\cos\theta > 0$, $\tan\theta > 0$

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \left(\frac{1}{3}\right)^2 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$

$\cos^2\theta = 8/9 \Rightarrow \cos\theta = \sqrt{8}/3$



$\boxed{y = \sin^{-1} x}$

1^{st} quad.

4^{th} quad.

FIGURE 4
$y = \sin^{-1} x = \arcsin x$

$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/3}{\sqrt{8}/3} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$

Domain $= [-1, 1]$, Range $= [-\pi/2, \pi/2]$

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$
$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

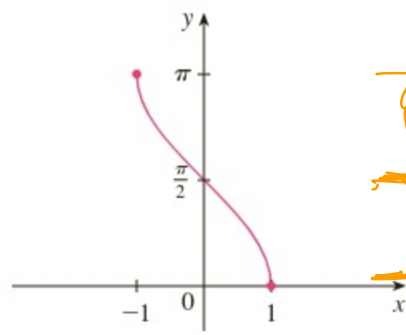
◇ **Inverse Cosine Function**. *Notation*: $\cos^{-1}(x)$ or $\arccos(x)$

$$\boxed{\cos^{-1} x = \theta \iff \cos\theta = x \quad \text{for } 0 \le \theta \le \pi}$$

Domain $= [-1, 1]$

Range $= [0, \pi]$

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leqslant x \leqslant \pi$$
$$\cos(\cos^{-1} x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$



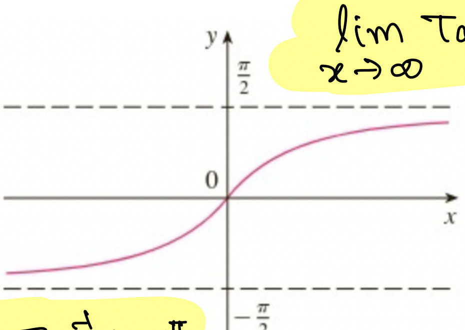
2^{nd} quadrant

1^{st} quadrant

◇ **Inverse Tangent Function**. *Notation*: $\tan -1(x)$ or $\arctan(x)$

$$\boxed{\tan^{-1} x = \theta \iff \tan\theta = x \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}}$$

$\lim_{x \to \infty} \tan^{-1} x = \pi/2$



1^{st} quad.

4^{th} quad.

$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

Domain $= \mathbb{R}$

Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

not in range

$x > 0 \Rightarrow \tan^{-1} x > 0$ (lies in 1^{st} quad)

$x < 0 \Rightarrow \tan^{-1} x < 0$ (lies in 4^{th} quad)

Example 2: Evaluate (a) $\cos^{-1}(-1)$ and (b) $\arctan(\sqrt{3})$.

(a) $\cos^{-1}(-1) = \pi$ since $\cos\pi = -1$

(b) $\arctan(\sqrt{3}) = \theta$ for which $\tan\theta = \sqrt{3}$

Example 3: Simplify the expression $\cos\left(\tan^{-1}(x)\right)$

$$\cos(\underbrace{\tan^{-1} x}_{\theta}) = \cos\theta$$

$x > 0 \Rightarrow \theta$ is in 1^{st} quad.
$x < 0 \Rightarrow \theta$ is in 4^{th} quad.
In both cases $\cos\theta > 0$

$\theta = \tan^{-1} x \Rightarrow \tan\theta = x \Rightarrow$ Given $\tan\theta$ we want to find $\cos\theta$

$\Rightarrow \sec^2\theta = 1 + \tan^2\theta = 1 + x^2 \Rightarrow \cos^2\theta = \dfrac{1}{\sec^2\theta} = \dfrac{1}{1+x^2}$

$\Rightarrow \cos(\tan^{-1} x) = \dfrac{\pm 1}{\sqrt{1+x^2}}$ (reject -ve sign) $\Rightarrow \cos(\tan^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$

Derivative and Integral Formulas Involving Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \sin^{-1} x + \cos^{-1} x = \pi/2$$

Example 4: Differentiate

(a) $H(x) = 2\tan^{-1}(x) + \arcsin(2x^2) + \cos^{-1}\left(\tan x\right)$

$$H'(x) = 2\left[\tan^{-1} x\right]' + \left[\arcsin(2x^2)\right]' + \left[\cos^{-1}(\tan x)\right]'$$

$$= \frac{2}{1+x^2} + \frac{1}{\sqrt{1-(2x^2)^2}} \cdot (2x^2)' + \frac{-1}{\sqrt{1-\tan^2 x}} \cdot (\tan x)'$$

$$\Rightarrow H'(x) = \frac{2}{1+x^2} + \frac{4x}{\sqrt{1-4x^4}} - \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}$$

chain rule: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

(b) $f(x) = x\arctan(\sqrt{x})$

Product rule

$$f'(x) = [x]' \arctan(\sqrt{x}) + x\left[\arctan(\sqrt{x})\right]'$$

$$= \arctan(\sqrt{x}) + x \cdot \frac{1}{1+(\sqrt{x})^2} \cdot \left[\sqrt{x}\right]'$$

$$= \arctan(\sqrt{x}) + \frac{x}{1+x} \cdot \frac{1}{2\sqrt{x}}$$

Example 5: Evaluate

$$= \arctan(\sqrt{x}) + \frac{\sqrt{x}}{2(1+x)}$$

(a) $\displaystyle\int \frac{1}{15\sqrt{1-x^2}}\,dx$

$$= \frac{1}{15}\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{15}\sin^{-1}x + C$$

(b) $\displaystyle\int \frac{3}{1+x^2}\,dx$

$$= 3\int \frac{1}{1+x^2}\,dx = 3\,\mathrm{Tan}^{-1}x + C$$

(c) $\displaystyle\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

$$u = \mathrm{Tan}\,x \Rightarrow \frac{du}{dx} = \sec^2 x \Rightarrow du = \underbrace{\sec^2 x}\cdot\underbrace{dx}$$

$$I = \int \frac{\sec^2 x}{\sqrt{1-\mathrm{Tan}^2 x}}\,dx = \int \frac{1}{\sqrt{1-\underbrace{\mathrm{Tan}^2 x}_{u^2}}}\,\underbrace{\sec^2 x\,dx}_{du}$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C$$

$$= \sin^{-1}(\mathrm{Tan}\,x) + C$$

(d) $\int_0^1 \frac{x}{1+x^4}\,dx$. **Note**: Evaluate all expressions into real numbers for your final answer.

$$\Rightarrow u = x^2 \qquad \Rightarrow \frac{du}{dx} = 2x \qquad \Rightarrow du = 2x\,dx$$

$$\Rightarrow \frac{1}{2}\,du = x \cdot dx$$

$$I = \int_0^1 \frac{x}{1+x^4}\,dx = \int_0^1 \underbrace{\frac{1}{1+x^4}}_{1+(x^2)^2} \cdot \underbrace{x\,dx}_{\frac{1}{2}du}$$

$$= \int_{0^2}^{1^2} \frac{1}{1+u^2} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\,\mathrm{Tan}^{-1} u \Big|_0^1 = \frac{1}{2}\,\mathrm{Tan}^{-1} 1 - \frac{1}{2}\,\mathrm{Tan}^{-1} 0$$

$$= \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2} \cdot 0$$

$$= \frac{\pi}{8}$$