■ **Section 6.7** exercises, *page 489*: #1, 3, <u>7</u>, <u>8</u>, 9, 30, 31, 32, 33, 36, 37, 38, 59, 60, 61, 62, 63, 64.

## SUMMARY

- Definitions of Hyperbolic Functions and their graphs
- Some indentities
- Derivatives of Hyperbolic Functions. Hence, we get some more integral formulas.

Certain even and odd combinations of the exponential functions $e^x$ and $e^{-x}$ arise so frequently in mathematics and its applications that they deserve to be given special names. These are the ***Hyperbolic Functions***. In many ways, the hyperbolic functions are analogous to the trigonometric functions.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

## Graphs of Hyperbolic Functions

Range $= \mathbb{R}$

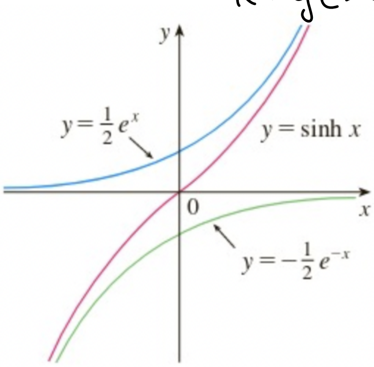


**FIGURE 1**
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

Range $= [1, \infty)$

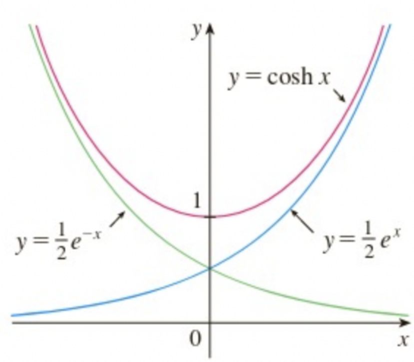


**FIGURE 2**
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

Range $= (-1, 1)$

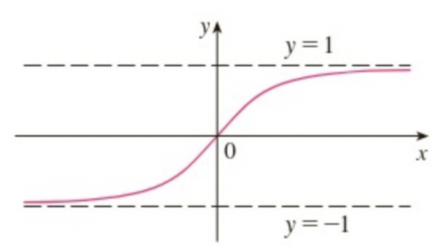


**FIGURE 3**
$y = \tanh x$

$\lim_{x \to \infty} \tanh x = 1$

$\lim_{x \to -\infty} \tanh x = -1$

For all three, domain is $\mathbb{R}$.

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

**Hyperbolic Identities**

$$\sinh(-x) = -\sinh(x) \quad \text{(odd function)} \qquad \cosh(-x) = \cosh x \quad \text{(even function)}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Here are the derivative formulas of Hyperbolic Functions. Note that from these formulas, we also obtain integral formulas.

**Derivatives of Hyperbolic Functions**

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

**Inverse Hyperbolic Functions:** See textbook, page 486.

*Example 1*: Compute the derivative of $y = \tanh^5(x^5)$

$$y = [\operatorname{Tanh}(x^5)]^5 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left([\operatorname{Tanh}(x^5)]^5\right)$$

$$z = \operatorname{Tanh}(x^5) \Rightarrow \frac{dy}{dx} = \frac{d}{dz}(z^5) \cdot \frac{dz}{dx} = 5z^4 \cdot \frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{d}{dx}\left(\operatorname{Tanh}(x^5)\right) = \frac{d}{dx}\left(\operatorname{Tanh}(u)\right) = \frac{d}{du}\left(\operatorname{Tanh}(u)\right)\frac{du}{dx}$$

$$u = x^5 \Rightarrow \frac{du}{dx} = 5x^4 \qquad = \boxed{\operatorname{Sech}^2(u) \cdot 5x^4}$$

$$\Rightarrow \frac{dy}{dx} = 5z^4 \cdot \operatorname{sech}^2(u) \cdot 5x^4$$

$$= 25 \tanh^4(x^5) \cdot \operatorname{sech}^2(x^5)\, x^4$$

*Example 2*: Evaluate the integral

(a) $\displaystyle\int \frac{\sinh(\ln x)}{x}\,dx$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow du = \frac{1}{x} \cdot dx$$

$$I = \int \sinh(\underbrace{\ln x}_{u}) \cdot \underbrace{\frac{1}{x} \cdot dx}_{du} = \int \sinh(u) \cdot du$$

$$= \cosh(u) + C$$

$$= \cosh(\ln x) + C$$

(b) $\displaystyle\int \frac{\sinh x}{1+\cosh x}\,dx$

$$u = 1 + \cosh x \Rightarrow \frac{du}{dx} = \sinh x \Rightarrow du = \sinh x \cdot dx$$

$$\Rightarrow I = \int \frac{\sinh x}{1+\cosh x}\,dx = \int \frac{1}{\underbrace{1+\cosh x}_{u}} \cdot \underbrace{\sinh x \cdot dx}_{du}$$

$$= \int \frac{1}{u} \cdot du$$

$$= \ln|u| + C = \ln|1+\cosh x| + C$$

(c) **What about** $\displaystyle\int \frac{\sinh x}{1+\cosh^2 x}\,dx$?

$$u = \cosh x \Rightarrow \frac{du}{dx} = \sinh x \Rightarrow du = \sinh x \cdot dx$$

$$I = \int \frac{1}{1+\underset{u^2}{\cosh^2 x}} \cdot \underbrace{\sinh x \, dx}_{du}$$

$$= \int \frac{1}{1+u^2} \, du = \tan^{-1}(u) + C$$

$$= \tan^{-1}(\cosh x) + C$$