M16600 Lecture Notes

Section 6.8: Indeterminate Forms and L'Hospital's Rule

■ **Section 6.8** exercises, *page* : #9, 15, 19, 21, 27, 35, 37, 43, 47, 52, 53, 57, 59, 65. *Optional*: Practice more problems from #8 to #68.

GOALS: Use L'Hospital's Rule to compute the limit of the following ***indeterminate form***

- **Indeterminate Quotient**: $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$
- **Indeterminate Product**: $0 \cdot \infty$
- **Indeterminate Difference**: $\infty - \infty$
- **Indeterminate Power**: 0^0, ∞^0, 1^∞

The Intuition of a Limit Statement: $\lim_{x\to 1}(x^2+2) = 3$. This equation states that as x approaches 1 (from the left and the right side of 1), the values of x^2+2 approaches 3.

Some Notation:

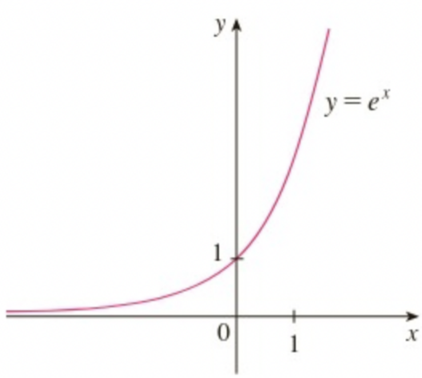
$x \to 1^+$ means x approaches 1 from the RIGHT, i.e., x is slightly BIGGER than 1 (e.g., $x = 1.01, 1.000012$, etc.)

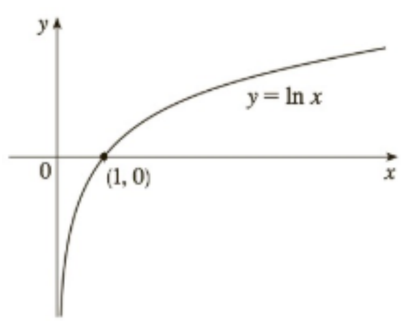
$x \to 1^-$ means x approaches 1 from the LEFT, i.e., x is a little SMALLER than 1 (e.g., $x = 0.99, 0.999999$, etc.)

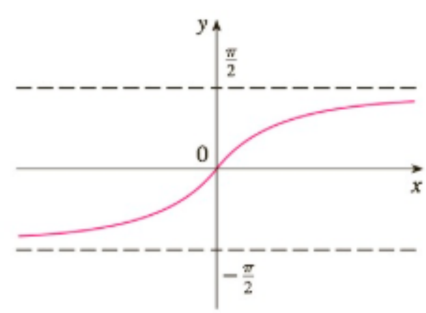
$x \to 1$ means x approaches 1 from both directions, left and right (i.e., x can take any values slightly less than or bigger than 1)

Warning: 1^- does NOT mean -1.

Limit Facts about e^x, $\ln x$, and $\arctan(x)$







$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to-\infty} e^x = 0$

$\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to 0^+} \ln x = -\infty$

$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$

$\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$

Computing Limits: The FIRST step in computing limit is what I call ***"direct substitution"*** (D.S.) Keep in mind, $x \to 1$ means x is very close to 1 but never equal 1.

After we do "direct substitution", we either get a **determinate form** or an **indeterminate form**.

Determinate Forms

- A real number $\to$ the limit is this real number
- $\dfrac{\text{a number}}{\pm\infty}$ $= 0$ Eg. $\frac{1}{\infty} = 0$, $\frac{100}{-\infty} = 0$, $\frac{\pi}{\infty} = 0$
- $\dfrac{\text{a nonzero number}}{0}$ $= \pm\infty$ Eg. $\frac{1}{0} = \infty$, $\frac{-1}{0} = -\infty$

Indeterminate Forms

- $\frac{0}{0} \to$ in section 1.6, we learn some algebra techniques to find the limit. In this section, we can apply L'Hospital's rule.
- $\frac{\pm\infty}{\pm\infty} \to$ in section 3.4, we learn a technique to solve this case. In this section, we can apply *L'Hospital's Rule* for this indeterminate form.
- $0 \cdot \infty \to$ rewrite as indeterminate quotient form then apply *L'Hospital's Rule.*
- $\infty - \infty \to$ rewrite as indeterminate quotient form then apply *L'Hospital's Rule.*
- $0^0,\ \infty^0,\ 1^\infty \to$ apply the tool of natural log then rewrite into indeterminate quotient form then apply *L'Hospital's Rule.*

L'Hosptital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a).

$\to$ g cannot be the constant 0 function

Suppose that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} \to \dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$. Then, by ***L'Hospital's Rule***, we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{1}$$

provide that the limit on the right side of the equation exists or is $\pm\infty$.

Note: L'Hospital's Rule also applies for $x \to a^+$, $x \to a^-$, or $x \to \pm\infty$.

Remark: We can apply L'Hospital more than one times if needed.

Examples: Evaluate the following limits. **Warning:** Don't blindly use L'Hospital's rule for every problem, see if it applies.

(a) $\lim_{x\to 1} \frac{\ln x}{x-1}$ $\overset{\text{D.S.}}{=\!=}$ $\frac{\ln 1}{1-1} = \frac{0}{0}$ (indeterminate)

$$= \lim_{x\to 1} \frac{(\ln x)'}{(x-1)'} \quad \text{(LH Rules)}$$

$$= \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} \overset{\text{D.S.}}{=\!=} \frac{1}{1} = 1$$

(b) $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$ $\overset{\text{D.S.}}{=\!=}$ $\frac{\ln\infty}{\sqrt[3]{\infty}} = \frac{\infty}{\infty}$ (indeterminate)

$$= \lim_{x\to\infty} \frac{(\ln x)'}{(\sqrt[3]{x})'} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{1/3-1}} = \lim_{x\to\infty} 3\cdot\frac{1}{x}\cdot\frac{1}{x^{-2/3}}$$

(LH Rules)

$$= \lim_{x\to\infty} 3\cdot\frac{1}{x}\cdot x^{2/3} = \lim_{x\to\infty} \frac{3}{x^{1-2/3}}$$

$$= \lim_{x\to\infty} \frac{3}{x^{1/3}} \overset{\text{D.S.}}{=\!=} \frac{3}{\infty} = 0$$

(c) $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$

$$\overset{\text{D.S.}}{=\!=} \frac{\sin\pi}{1-\cos\pi} = \frac{0}{1-(-1)} = \frac{0}{1+1} = 0$$

INCORRECT

$$\lim_{x\to\pi} \frac{\cos x}{\sin x} = \frac{\cos\pi}{\sin\pi} = \frac{-1}{0} = -\infty$$

(d) $\lim_{x\to\infty} \sqrt{x}e^{-x/2}$ $\underset{=}{\text{D.S.}}$ $\sqrt{\infty} \cdot e^{-\infty/2} = \infty \cdot 0$ (indeterminate)

$\infty \cdot \frac{0}{1} = \frac{\infty}{1/0} = \frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}} \overset{\text{D.S.}}{=} \frac{\sqrt{\infty}}{e^{\infty/2}} = \frac{\infty}{\infty}$$

(rewrite)

$$= \lim_{x\to\infty} \frac{[\sqrt{x}]'}{[e^{x/2}]'} = \lim_{x\to\infty} \frac{\cancel{\frac{1}{2}} x^{-1/2}}{\cancel{\frac{1}{2}} e^{x/2}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\, e^{x/2}}$$

$$= \frac{1}{\infty\cdot\infty} = \frac{1}{\infty} = 0$$

(e) $\lim_{x\to 0^+} x \ln x$

$$= 0 \cdot \ln 0^+ = 0 \cdot (-\infty) = -0\cdot\infty \quad \text{(indeterminate)}$$

rewrite $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{\text{D.S.}}{=} \frac{-\infty}{\infty}$ (indeterminate)

(LH rules) $= \lim_{x\to 0^+} \frac{[\ln x]'}{[1/x]'} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{1}{x}\cdot(-x^2)$

$$= \lim_{x\to 0^+} -x \overset{\text{D.S.}}{=} 0$$

(f) $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$

rewrite

$$\overset{\text{D.S.}}{=} \frac{1}{\cos \pi/2} - \frac{\sin \pi/2}{\cos \pi/2} = \frac{1}{0} - \frac{1}{0} = \infty - \infty$$

(indeterminate)

$$\lim_{x\to\frac{\pi}{2}^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x} \overset{\text{D.S.}}{=} \frac{1-\sin \pi/2}{\cos \pi/2}$$

(take common denominator)

$$= \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{[1-\sin x]'}{[\cos x]'} = \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} \overset{\text{D.S.}}{=} \frac{\cos \pi/2}{\sin \pi/2} = \frac{0}{1} = 0$$

(g) $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$ $\underline{\underline{DS}}$ $(1+0)^{\cot 0} = 1^{\infty}$

$\boxed{0^0 / \infty^0 / 1^{\infty}}$

$$L = \lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$$

$$\Rightarrow \ln L = \ln\left[\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}\right] = \lim_{x\to 0^+} \ln (1+\sin 4x)^{\cot x}$$

$$= \lim_{x\to 0^+} \underbrace{\cot x}_{=\frac{1}{\tan x}} \ln(1+\sin 4x) \underline{\underline{D.S.}} \cot 0 \ln(1+0) = \underbrace{\infty\cdot 0}_{\text{indeterminate}}$$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \underline{\underline{D.S.}} \frac{\ln(1+0)}{\tan 0} = \underbrace{\frac{0}{0}}_{\text{indeterminate}}$$

(LH rule) $$= \lim_{x\to 0^+} \frac{[\ln(1+\sin 4x)]'}{[\tan x]'} = \lim_{x\to 0^+} \frac{\left(\frac{1}{1+\sin 4x}\right)\cdot(\cos 4x)\cdot(4)}{\sec^2 x}$$

$$= \lim_{x\to 0^+} \frac{4\cos 4x}{\sec^2 x(1+\sin 4x)} \underline{\underline{D.S.}} \frac{4\cos 0}{\sec^2 0(1+\sin 0)} = 4$$

$$\Rightarrow \ln L = 4 \Rightarrow \boxed{L = e^4}$$

(h) $\lim_{x\to 0^+} x^x$ $\underline{\underline{DS}}$ 0^0

$$L = \lim_{x\to 0^+} x^x$$

$$\Rightarrow \ln L = \lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x\ln x = 0 \quad \text{(Problem (e))}$$

$$\Rightarrow \ln L = 0$$

$$\Rightarrow L = e^0 \Rightarrow \boxed{L = 1}$$