■ **Section 7.2** exercises, *page 524*: #1, 3, 7, 21, 23, 25, 13, 27, 17, 11, 29.

In this section, there are no new methods of integration. We mainly concern about **integrals that involve only trigonometric functions**, which we will call ***Trigonometric Integrals***.

Then main tools we are going to use to solve trigonometric integrals are

- The method of u-substitution
- Trigonometric identities

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad \sin(2x) = 2\sin x\,\cos x$$

$$\cos^2 x = \frac{1}{2}\big[1 + \cos(2x)\big] \qquad \sin^2 x = \frac{1}{2}\big[1 - \cos(2x)\big]$$

- Sometimes, we will need to do integration by parts

Example 1: Evaluate $\int \sin^5 x \cos^2 x\, dx$ $(m = 5,\ n = 2)$

$$= \int \sin^4 x \underbrace{\cos^2 x}_{u^2} \underbrace{\sin x\, dx}_{-du}$$

$u = \cos x \Rightarrow du = -\sin x\, dx$

$(\sin^2 x)^2 = (1 - u^2)^2$

$$= \int (1-u^2)^2\, u^2\, (-du) = -\int u^2 (u^2 - 1)^2\, du$$

$$\int \sin^m x \cos^n x\, dx = \begin{cases} \text{If } m \text{ is odd, then substitute } u = \cos x \\ \text{If } n \text{ is odd, then substitute } u = \sin x \end{cases}$$

Example 2: Find $\int \cos^3 x\,dx$ $\quad u = \sin x \Rightarrow du = \cos x\,dx$

$(m=0,\ n=3)$

$$I = \int \cos^2 x \cos x\,dx = \int \underbrace{(1-\sin^2 x)}_{1-u^2}\,\underbrace{\cos x\,dx}_{du}$$

$$= \int (1-u^2)\,du = u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

(*) what happens when both m and n are even?

Example 3: Evaluate $\int_0^{\pi} \sin^2 x\,dx$ $\quad \sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$I = \int \underbrace{\sin^2 x}\,dx = \int \underbrace{\frac{1}{2}(1-\cos 2x)}\,dx$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\underbrace{\int 1\,dx}_{x} - \frac{1}{2}\int \cos 2x\,dx$$

$z = 2x$

$\Rightarrow dz = 2\,dx$

$\Rightarrow \frac{1}{2}dz = dx$

$$\int \cos 2x\,dx = \int \cos z \left(\tfrac{1}{2}dz\right)$$

$$= \frac{1}{2}\int \cos z\,dz = \frac{1}{2}\sin z + C = \frac{1}{2}\sin 2x + C$$

$$I = \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + C$$

$$\Rightarrow \int_0^{\pi} \sin^2 x\,dx = \frac{1}{2}x\Big|_0^{\pi} - \frac{1}{4}\sin 2x\Big|_0^{\pi} = \frac{\pi}{2} - \frac{1}{4}(\underbrace{\sin 2\pi}_{=0} - \underbrace{\sin 0}_{=0}) = \frac{\pi}{2}$$

Example 4: Find $\int \tan^6 x \sec^4 x \, dx$ $= \int \underbrace{\tan^6 x}_{u^6} \, \underbrace{\sec^2 x}_{1+u^2} \cdot \underbrace{\sec^2 x \, dx}_{du}$

$\boxed{u = \tan x} \Rightarrow \frac{du}{dx} = \sec^2 x$

$\Rightarrow du = \sec^2 x \, dx$

works as long as Power of $\sec x$ is even

$$= \int u^6 (1+u^2) \, du$$

$$= \int (u^6 + u^8) \, du$$

$$= \int u^6 \, du + \int u^8 \, du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

Example 5: Find $\int \tan^5 \theta \sec^7 \theta \, d\theta$ $= \int \underbrace{\tan^4 \theta}_{} \, \underbrace{\sec^6 \theta}_{u^6} \, \underbrace{\tan\theta \sec\theta \, d\theta}_{du}$

$\boxed{u = \sec \theta}$

$\Rightarrow du = \sec\theta \tan\theta \, d\theta$

works as long as the Power of $\tan\theta$ is odd.

$$(\tan^2\theta)^2 = (\sec^2\theta - 1)^2$$

$$= \int (u^2-1)^2 u^6 \, du$$

$$= \int u^6 [u^4 - 2u^2 + 1] \, du$$

$$= \int (u^{10} - 2u^8 + u^6) \, du$$

$$= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C$$

$$= \frac{1}{11}\sec^{11}\theta - \frac{2}{9}\sec^9\theta + \frac{1}{7}\sec^7\theta + C$$

Extra Examples: $\int \tan^m x \sec^n x \, dx = \begin{cases} \text{If } m \text{ is odd, use } u = \sec x \\ \text{If } n \text{ is even, use } u = \tan x \end{cases}$

- $\int \tan^3 x \, dx$ (Example 7, textbook, page 523).
- $\int \sec^3 x \, dx$ (Example 8, textbook, page 523).
- $\int \sin(4x)\cos(5x) \, dx$ (Example 9, textbook, page 524)

Example 6: Compute $\int \sin(2x)\cos^2 x \, dx$.

$\sin(2x) = 2 \sin x \cos x$

$$= \int 2 \sin x \cos x \cos^2 x \, dx$$

$$= 2 \int \sin x \cos^3 x \, dx$$