■ **Section 7.3** exercises, *page 531*: #1, 2, 5, 6, 8, 12, 14, 9, 22, 17, 11.

Trigonometric Substitution is a new method which oftentimes are useful in solving integrals that involves the following radicals. We will also give the appropriate trig substitution for each type of radical:

$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

$\sqrt{4-x^2} \rightarrow x = 2\sin\theta$

$\sqrt{3+x^2} \rightarrow x = \sqrt{3}\tan\theta$

$\sqrt{x^2-1} \rightarrow x = \sec\theta$

We might need these two formulas for integrals in this section:

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \underbrace{\int \sec x\, dx = \ln|\sec x + \tan x| + C}_{\text{HW}}$$

Example 1: Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

Step 1

$x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin\theta)^2}{\sqrt{9-(3\sin\theta)^2}} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta\, d\theta = \int \frac{9\sin^2\theta}{\sqrt{9(1-\sin^2\theta)}} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{9\sin^2\theta}{\sqrt{9\cos^2\theta}} \cdot 3\cos\theta\, d\theta = \int \frac{9\sin^2\theta}{\cancel{3\cos\theta}} \cdot \cancel{3\cos\theta}\, d\theta$$

$$= \int 9 \sin^2\theta \, d\theta = 9 \int \sin^2\theta \, d\theta$$

Step2

$$= \frac{9}{2} \int (1 - \cos 2\theta) \, d\theta = \frac{9}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right] + C$$

$$= \frac{9}{2} \left[\theta - \frac{1}{2} \cdot 2 \sin\theta \cos\theta \right] + C$$

$x = 3 \sin\theta \Rightarrow \sin\theta = \frac{x}{3}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$$= \frac{9}{2} \left[\theta - \sin\theta \cos\theta \right] + C$$

Example 2: Compute $\displaystyle\int_2^3 \frac{1}{\sqrt{x^2 - 1}} \, dx$

Step3

$\cos^2\theta = 1 - \frac{x^2}{9} = \frac{9 - x^2}{9}$

$\Rightarrow \cos\theta = \frac{\sqrt{9 - x^2}}{3}$

$$= \frac{9}{2} \left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3} \frac{\sqrt{9 - x^2}}{3} \right] + C$$

$$= \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2} \sqrt{9 - x^2} + C$$

$$\int \frac{1}{\sqrt{x^2 - 1}} \, dx$$

$\boxed{x = \sec\theta} \Rightarrow dx = \sec\theta \tan\theta \, d\theta$

$$\int \frac{1}{\sqrt{x^2 - 1}} \, dx = \int \frac{1}{\sqrt{\sec^2\theta - 1}} \cdot \sec\theta \tan\theta \, d\theta$$

$$= \int \frac{1}{\sqrt{\tan^2\theta}} \cdot \sec\theta \tan\theta \, d\theta$$

$$= \int \frac{1}{\cancel{\tan\theta}} \cdot \sec\theta \, \cancel{\tan\theta} \, d\theta = \int \sec\theta \, d\theta = \ln \left| \underset{\uparrow\, x}{\sec\theta} + \underbrace{\tan\theta}_{?} \right| + C$$

$\sec\theta = x$, $\tan^2\theta = \sec^2\theta - 1 = x^2 - 1 \Rightarrow \tan\theta = \sqrt{x^2 - 1}$

$$\int \frac{1}{\sqrt{x^2 - 1}} \, dx = \ln \left| x + \sqrt{x^2 - 1} \right| + C$$

$$\Rightarrow \int_2^3 \frac{1}{\sqrt{x^2 - 1}} \, dx = \ln \left| 3 + \sqrt{9 - 1} \right| - \ln \left| 2 + \sqrt{2^2 - 1} \right| = \ln \left(\frac{3 + \sqrt{8}}{2 + \sqrt{3}} \right)$$

Example 3: Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

$x = 2\tan\theta$

$\Rightarrow dx = 2\sec^2\theta\, d\theta$

Step 1

$$I = \int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{(2\tan\theta)^2\sqrt{(2\tan\theta)^2+4}}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} \cdot 2\sec^2\theta\, d\theta$$

$$= \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4(\tan^2\theta+1)}}\,d\theta = \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\sec^2\theta}}\,d\theta$$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{4\tan^2\theta(\cancel{2}\,\cancel{\sec\theta})}\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

Step 2

$$I = \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$

$\Rightarrow du = \cos\theta\, d\theta$

$$= \frac{1}{4}\int \frac{1}{u^2}\,du = \frac{1}{4}\int u^{-2}\,du = \frac{1}{4}\,\frac{u^{-2+1}}{-2+1} + C$$

$$= \frac{-1}{4u} + C$$

Step 3

$$I = \frac{-1}{4u} + C = \frac{-1}{4\sin\theta} + C$$

$x = 2\tan\theta \rightsquigarrow$ want to find $\sin\theta$

$$\sin\theta = \tan\theta\cos\theta = \frac{x}{2}\cos\theta$$

$$\tan\theta = \frac{x}{2} \Rightarrow \sec^2\theta = 1 + \frac{x^2}{2^2} = 1 + \frac{x^2}{4} = \frac{4+x^2}{4}$$

$$\Rightarrow \sec\theta = \frac{\sqrt{x^2+4}}{2} \Rightarrow \cos\theta = \frac{2}{\sqrt{x^2+4}}$$

$$\Rightarrow \sin\theta = \frac{x}{2} \cdot \frac{2}{\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+4}}$$

$$I = \frac{-1}{4\sin\theta} + C = \frac{-1}{4\left(\frac{x}{\sqrt{x^2+4}}\right)} + C = \frac{-\sqrt{x^2+4}}{4x} + C$$