■ **Section 7.4** exercises, *page 541*: #9, 12, 19, 23, 24, 10, 11, 20, 25.

Terminologies:

- **Rational Function:** a ratio of polynomials
- **Partial Fractions Decomposition:** is the technique of decomposing rational function into a combination of simpler fractions

  E.g., $\dfrac{x+5}{x^2+x-2} = \dfrac{2}{x-1} - \dfrac{1}{x+2}$
- **Integration by Partial Fractions:** is a method of integrating certain types of rational functions by first decomposing the rational function into simpler fractions then integrate.

  E.g., $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx = 2\ln|x-1| - \ln|x+2| + C$

In order to perform the method of Integration by Partial Fractions, we need to be able to do these three processes:

1. Writing out the form of the partial fractions decomposition
2. Finding the values of the coefficients
3. Doing a u-substitution

Example 1 (Process 1): Write out the form of the partial fractions decomposition of the functions

Step 1: If the (highest degree of the numerator) is $\geq$ the (highest degree of the denominator), do long division
Step 2: Factor the denominator completely
Step 3: Treat ***Linear Factor*** (highest degree is 1) and ***Quadratic Factor*** (highest degree is 2) differently
Step 4: Take care of ***the multiplicity*** of each factor accordingly

(a) $\dfrac{x+5}{x^2+x-2} = \dfrac{x+5}{(x-1)(x+2)}$

$x^2+x-2 = x^2-x+2x-2$

$= x(x-1)+2(x-1) = (x-1)(x+2)$

$\dfrac{x+5}{(x-1)(x+2)} = \dfrac{a}{x-1} + \dfrac{b}{x+2}$

(b) $\dfrac{x^3-x+1}{x(x+4)^3(x^2+4)}$

$= \dfrac{a}{x} + \dfrac{b}{x+4} + \dfrac{c}{(x+4)^2} + \dfrac{d}{(x+4)^3} + \dfrac{ex+f}{x^2+4}$

(c) $\dfrac{x^3+x^2+1}{x^2(x-1)(x^2+x+1)(x^2+1)^2}$

$$= \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1} + \frac{dx+e}{x^2+x+1} + \frac{fx+g}{x^2+1} + \frac{hx+i}{(x^2+1)^2}$$

Example 2 (Processes 1 and 2): Write out the form of the partial fraction decomposition of the functions then find the values of the coefficients

(a) $\dfrac{x+5}{(x-1)(x+2)} = \dfrac{a}{x-1} + \dfrac{b}{x+2}$ $\Big] \times (x-1)(x+2)$

$$\cancel{(x-1)(x+2)} \cdot \frac{x+5}{\cancel{(x-1)(x+2)}} = \cancel{(x-1)}(x+2)\cdot\frac{a}{\cancel{x-1}} + (x-1)\cancel{(x+2)}\cdot\frac{b}{\cancel{x+2}}$$

$$\Rightarrow x+5 = a(x+2) + b(x-1)$$
$$= ax + 2a + bx - b$$
$$= (a+b)x + 2a - b$$

$\Rightarrow a+b=1$ and $2a-b=5$

$$a+\cancel{b}=1$$
$$2a-\cancel{b}=5$$
$$3a = 6 \Rightarrow a=2$$

$$2+b=1 \Rightarrow b=-1$$

$a=2$, $b=-1$

Alternatively

$x+5 = a(x+2)+b(x-1)$

Put $x=1$

$1+5 = a(1+2) + \cancel{b(1-1)} \rightarrow 0$

$6 = 3a \Rightarrow \boxed{a=2}$

Put $x=-2$

$-2+5 = \cancel{a(-2+2)} \rightarrow 0 + b(-2-1)$

$3 = -3b \Rightarrow \boxed{b=-1}$

$$\frac{x+5}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{-1}{x+2}$$

(b) $$\left[\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{a}{x} + \frac{b}{2x-1} + \frac{c}{x+2}\right] \times x(2x-1)(x+2)$$

$$x^2+2x-1 = \cancel{x}(2x-1)(x+2)\frac{a}{\cancel{x}} + x\cancel{(2x-1)}(x+2)\frac{b}{\cancel{2x-1}} + x(2x-1)\cancel{(x+2)}\frac{c}{\cancel{x+2}}$$

$$x^2+2x-1 = a(2x-1)(x+2) + bx(x+2) + cx(2x-1)$$

$$x^2+2x-1 = a(2x^2+3x-2) + b(x^2+2x) + c(2x^2-x)$$

$$= (2a+b+2c)x^2 + (3a+2b-c)x - 2a$$

$2a+b+2c=1$

$3a+2b-c=2$

$-2a=-1 \Rightarrow \boxed{a=\frac{1}{2}}$

$\cancel{1}+b+2c=\cancel{1} \Rightarrow b=-2c$

$3\left(\frac{1}{2}\right)+2(-2c)-c=2$

$\frac{3}{2}-4c-c=2 \Rightarrow -5c=\frac{1}{2}$

$\Rightarrow \boxed{c=\frac{-1}{10}}$

$\Rightarrow \boxed{b=-2c=\frac{1}{5}}$

Alternatively

Put $x=0$:

$-1 = a(-1)(2) \Rightarrow -1=-2a$

$\Rightarrow \boxed{a=\frac{1}{2}}$

Put $x=\frac{1}{2}$:

$\left(\frac{1}{2}\right)^2+2\left(\frac{1}{2}\right)-1 = b\,\frac{1}{2}\left(\frac{1}{2}+2\right)$

$\frac{1}{4}+1-1 = b\cdot\frac{1}{2}\cdot\frac{5}{2}$

$\frac{1}{4} = \frac{5}{4}\cdot b \Rightarrow \boxed{b=\frac{1}{5}}$

Put $x=-2$:

$(-2)^2+2(-2)-1 = c(-2)(-4-1)$

$4-4-1 = 10c \Rightarrow \boxed{c=\frac{-1}{10}}$

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{1}{2}\cdot\frac{1}{x} + \frac{1}{5}\cdot\frac{1}{2x-1} - \frac{1}{10}\cdot\frac{1}{x+2}$$

$$\int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx = \frac{1}{2}\int\frac{1}{x}\,dx + \frac{1}{5}\int\frac{1}{2x-1}\,dx - \frac{1}{10}\int\frac{1}{x+2}\,dx$$

$$= \boxed{\frac{1}{2}\ln|x| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C}$$

Example 3 (Process 3): Evaluate

1. $\int \frac{1}{x+2}\,dx$

$= \ln|x+2| + C$

2. $\int \frac{2}{x-1}\,dx$

$= 2\ln|x-1| + C$

3. $\int \frac{1}{5}\frac{1}{2x-1}\,dx$

$= \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| + C$

$u = 2x-1$

4. $\int \frac{2}{(x-1)^2}\,dx \quad = 2\int \frac{1}{(x-1)^2}\,dx$

$u = x-1$

$du = dx$

$= 2\int \frac{1}{u^2}\,du$

$= 2\,\frac{u^{-2+1}}{-2+1} + C = 2\,\frac{u^{-1}}{-1} + C = \frac{-2}{x-1} + C$

Example 4: Evaluate $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$

$$\left[\frac{5x+1}{(2x+1)(x-1)} = \frac{a}{2x+1} + \frac{b}{x-1}\right] \times (2x+1)(x-1)$$

$$\Rightarrow \quad 5x+1 = a(x-1) + b(2x+1)$$

Put $x=1$:

$$6 = b(3)$$

$$\Rightarrow b = 2$$

Put $x = -\frac{1}{2}$:

$$-\frac{5}{2}+1 = a\left(-\frac{1}{2}-1\right)$$

$$-\frac{3}{2} = a\left(-\frac{3}{2}\right) \Rightarrow a = 1$$

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{1}{2x+1} + \frac{2}{x-1}$$

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \frac{1}{2x+1}\,dx + 2\int \frac{1}{x-1}\,dx$$

$$= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$$

Example 5: Evaluate $\displaystyle\int \frac{4x}{(x-1)^2(x+1)}\,dx$

$$\left[\frac{4x}{(x-1)^2(x+1)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1}\right] \times (x-1)^2(x+1)$$

$$4x = a(x-1)(x+1) + b(x+1) + c(x-1)^2$$

Put $x=1$:

$4 = 2b$

$\Rightarrow b = 2$

Put $x=-1$:

$-4 = c(-1-1)^2$

$-4 = 4c \Rightarrow c = -1$

Put $x=0$:

$0 = -a + b + c$

$0 = -a + 2 + (-1)$

$\Rightarrow a = 1$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx - \int \frac{1}{x+1}\,dx$$

$u = x-1$

$$= \ln|x-1| + 2\int \frac{1}{u^2}\,du - \ln|x+1| + C$$

$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$$\frac{u^{-2+1}}{-2+1}$$

It is useful to remember this integral formula

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \tfrac{1}{a}\tan^{-1}\left(\tfrac{x}{a}\right)+C}$$

When $a=1$, the above formula becomes one we already know $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x)+C$.

Example 6: Evaluate $\displaystyle\int \frac{2x^2-x+1}{x^3+x}\,dx$

$$\left[\frac{2x^2-x+1}{x^3+x} = \frac{2x^2-x+1}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1}\right] x(x^2+1)$$

$$\Rightarrow 2x^2-x+1 = a(x^2+1) + (bx+c)x$$

Put $x=0$:

$1 = a(0^2+1)$

$\Rightarrow a=1$

Put $x=1$:

$2-1+1 = a(2)+b+c$

$2 = 2a+b+c$

Put $x=-1$:

$2+1+1 = a(2)+(-b+c)(-1)$

$4 = 2a+b-c$

$b+c=0$

$b-c=2$ $\Rightarrow 2b=2 \Rightarrow b=1$

$\Rightarrow c=-1$

$$\frac{2x^2-x+1}{x^3+x} = \frac{1}{x} + \frac{x-1}{x^2+1}$$

$$= \frac{1}{x} + \frac{x-1}{x^2+1}$$

$$\int \frac{2x^2-x+1}{x^3+x}\,dx = \underbrace{\int \frac{1}{x}\,dx}_{\ln|x|} + \int \frac{x-1}{x^2+1}$$

$$\int \frac{x-1}{x^2+1} = \int \frac{x}{x^2+1}\,dx - \underbrace{\int \frac{1}{x^2+1}\,dx}_{\tan^{-1}(x)}$$

$u = x^2+1$

$\Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+1| + C$$

$$\int \frac{2x^2-x+1}{x^3+x}\,dx = \int \frac{1}{x}\,dx + \int \frac{x-1}{x^2+1}$$

$$= \ln|x| + \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+1| - \arctan(x) + C$$