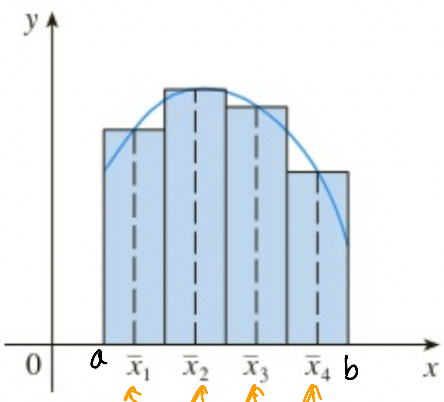
■ **Section 7.7** exercise: see the two bullets below

- Use (a) the Trapezoidal Rule, (b) The Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places.)
$$\int_1^3 \sqrt{x^3-1}\,dx, \qquad n = 6$$
- *Optional exercises*: section 7.7, page 654, # 5, 16.

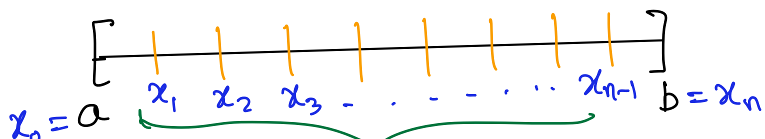
There are situations where it is difficult, or even impossible, to compute $\int_a^b f(x)\,dx$. Other times, when a function is determined from a scientific experiment through instrument readings or collect data, there may be no formula for the function. Therefore, it is needful to have methods for approximating definite integrals.

⋆ ***The Midpoint Rule*** *for approximating definite integrals*



(c) Midpoint approximation

midpoints:



n subintervals of equal length

Total length $= b - a$

$$\Delta x = \frac{b-a}{n}$$

(length of each subinterval)

$$x_i = a + i(\Delta x)$$

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} = \frac{1}{2}\left(a + (i-1)\Delta x + a + i(\Delta x)\right)$$

$$= \frac{1}{2}\left(2a + (2i-1)\Delta x\right)$$

$$\Rightarrow \bar{x}_i = a + \left(i - \frac{1}{2}\right)\Delta x$$

$$\int_a^b f(x)\,dx \simeq \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right]$$

★ ***The Trapezoidal Rule*** *for approximating definite integrals*

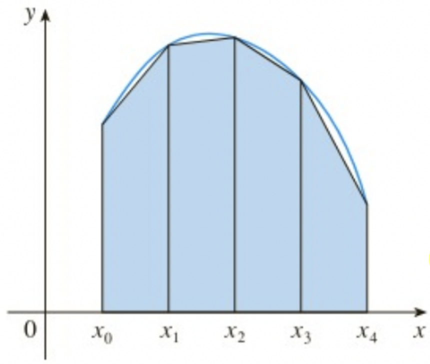


FIGURE 2
Trapezoidal approximation

$n =$ the number of subintervals or the number of trapezoids

$\Delta x = \dfrac{b-a}{n} =$ the height of each trapezoid

$T_n =$ the area of n trapezoids

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$\int_a^b f(x)\,dx \approx T_n$$

Area of i^{th} trapezoid $= \frac{\Delta x}{2}\left[f(x_{i-1}) + f(x_i)\right]$

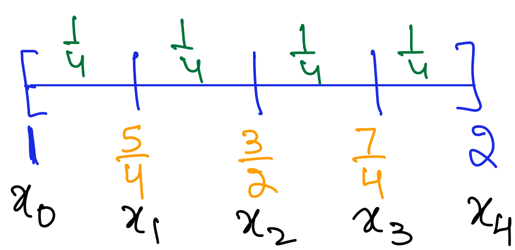
$\boxed{1 \quad 2 \quad 2 \quad \cdots \quad 2 \quad 1}$

Example 1: Use (a) the Midpoint Rule and (b) the Trapezoidal Rule with $n = 4$ to approximate the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

$a = 1, \; b = 2, \; n = 4 \Rightarrow \Delta x = \frac{2-1}{4} = \frac{1}{4}$

$x_0 = 1, \; x_1 = \frac{5}{4}, \; x_2 = \frac{3}{2},$

$x_3 = \frac{7}{4}, \; x_4 = 2$



$\overline{x_1} = \frac{x_0 + x_1}{2} = \frac{9}{8}, \qquad \overline{x_2} = \frac{x_1 + x_2}{2} = \frac{11}{8}$

$\overline{x_3} = \frac{x_2 + x_3}{2} = \frac{13}{8}, \qquad \overline{x_4} = \frac{x_3 + x_4}{2} = \frac{15}{8}$

(a) $\displaystyle\int_1^2 \frac{1}{x}\,dx = \frac{1}{4}\left[f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right]$

$= \frac{1}{4}\left[\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right]$

$= 2\left[\frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}\right] = 0.69$

(b) $\int_1^2 \frac{1}{x}\,dx = \frac{\Delta x}{2}\left[f(1) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2)\right]$

$$= \frac{1}{8}\left[1 + 2\cdot\frac{4}{5} + 2\cdot\frac{2}{3} + 2\cdot\frac{4}{7} + \frac{1}{2}\right]$$

$$= \frac{1}{8}\left[1 + \frac{8}{5} + \frac{4}{3} + \frac{8}{7} + \frac{1}{2}\right] = 0.697$$

$\ln 2 = 0.69347$

⋆ ***The Simpson's Rule*** *for approximating definite integrals*

Another rule for approximate integration results from using parabolas instead of straight line segments to approximate the curve.

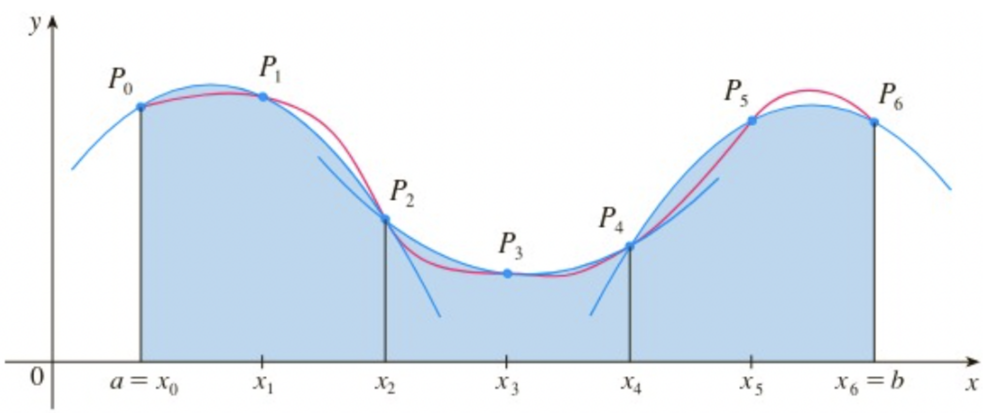


FIGURE 7

$n =$ the number of subintervals. n must be **even** for Simpson's Rule.

$\Delta x = \dfrac{b-a}{n} =$ the length of each subinterval

$S_n =$ the area under the parabolas by using Simpson's Rule

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Note the pattern of coefficients: 1, 4, 2, 4, 2, 4, 2,. . . , 4, 2, 4, 1.

$$\int_a^b f(x)\,dx \approx S_n$$

You can read the discussion on page 559–560 of the textbook to see how the formula for S_n is derived.

Example 2: Use Simpson's rule with $n = 8$ to approximate $\int_0^1 e^{x^2}\,dx$.

$a=0,\ b=1,\quad n=8 \Rightarrow \Delta x = \frac{1-0}{8} = \frac{1}{8},\quad x_i = 0 + i\left(\frac{1}{8}\right) = i/8$

$x_0=0,\ x_1=\frac{1}{8},\ x_2=\frac{1}{4},\ x_3=\frac{3}{8},\ x_4=\frac{1}{2},$

$x_5=\frac{5}{8},\ x_6=\frac{3}{4},\ x_7=\frac{7}{8},\ x_8=1.$

$$\int_0^1 e^{x^2}\,dx = \frac{1/8}{3}\left[e^{0^2} + 4e^{1/64} + 2e^{1/16} + 4e^{9/64} + 2e^{1/4} + 4e^{25/64} + 2e^{9/16} + 4e^{49/64} + e^{1}\right]$$

$$= \frac{1}{24}\left[1 + 4e^{1/64} + 2e^{1/16} + 4e^{9/64} + 2e^{1/4} + 4e^{25/64} + 2e^{9/16} + 4e^{49/64} + e\right]$$

Example 1 (Simpson's rule) $n=4$, $\int_1^2 \frac{1}{x}\,dx$

$$\int_1^2 \frac{1}{x}\,dx = \frac{\Delta x}{3}\left[f(1) + 4f\left(\tfrac{5}{4}\right) + 2f\left(\tfrac{3}{2}\right) + 4f\left(\tfrac{7}{4}\right) + f(2)\right]$$

$$= \frac{1}{12}\left[1 + 4\cdot\frac{4}{5} + 2\cdot\frac{2}{3} + 4\cdot\frac{4}{7} + \frac{1}{2}\right]$$

$$= \frac{1}{12}\left[1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2}\right] = 0.69325$$

$\ln 2 = 0.69347$