M16600 Lecture Notes
Section 7.8: Improper Integrals

■ **Section 7.8** textbook exercises, *page 574*: #2, 5, 7, 9, 11, 13, 19, 21, 27, 29, 31, 33.

GOALS

- Compute **improper integrals** of type I. E.g., $\displaystyle\int_1^\infty \frac{1}{x}\,dx$.
- Compute **improper integrals** of type II. E.g., $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

A definite integral $\displaystyle\int_a^b f(x)\,dx$ that we've encountered so far satisfies both of these conditions:

(i) The interval $[a, b]$ is finite and

(ii) The integrand $f(x)$ is continuous on $[a, b]$

If either one of the two conditions above fails, we say the definite integral to be ***improper***. Here are some examples of improper integrals

- ***Improper Integrals of Type I*** (condition (i) fails):

$$\int_1^\infty \frac{1}{x}\,dx, \qquad \int_{-\infty}^0 xe^x\,dx, \qquad \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx.$$

- ***Improper Integrals of Type II*** (condition (ii) fails):

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx, \qquad \int_0^1 \ln x\,dx, \qquad \int_{-1}^0 \frac{3}{x^3}\,dx, \qquad \int_0^3 \frac{1}{x-1}\,dx.$$

How to Compute Improper Integrals of Type I: Rewrite the integrals as follows:

- $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\left[\int_a^t f(x)\,dx\right]$
- $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\left[\int_t^b f(x)\,dx\right]$
- $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$ where c is a constant

Definitions:

· The improper integral is ***convergent*** if the limit = a finite number (i.e., the limit exists)

· The improper integral is ***divergent*** if the limit = $\pm\infty$ or the limit does not exist.

Example 1: Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$$\int_1^t \frac{1}{x}\,dx = \ln|x|\Big|_1^t = \ln t - \ln 1 = \ln t$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t = \infty$$

$$\Rightarrow \int_1^\infty \frac{1}{x}\,dx \text{ is divergent}$$

(b) $\displaystyle\int_{-\infty}^0 xe^x\,dx$

$$= \lim_{t\to-\infty} \int_t^0 xe^x\,dx$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$$

$u = x \Rightarrow du = dx$

$dv = e^x \Rightarrow v = e^x$

$$\int_t^0 xe^x\,dx = \left(xe^x - e^x\right)\Big|_t^0 = \left(0e^0 - e^0\right) - \left(te^t - e^t\right)$$
$$= -1 - te^t + e^t$$

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}\left(-1 - te^t + e^t\right) = -1 - \underbrace{\lim_{t\to-\infty} te^t}_{-\infty\cdot 0} + \underbrace{\lim_{t\to-\infty} e^t}_{0}$$

$$\lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{-e^{-t}} = \lim_{t\to-\infty} -e^t = 0$$

$\left(\frac{\infty}{\infty}\right)$

$$\int_{-\infty}^0 xe^x\,dx = -1 - 0 + 0 = -1$$

(c) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

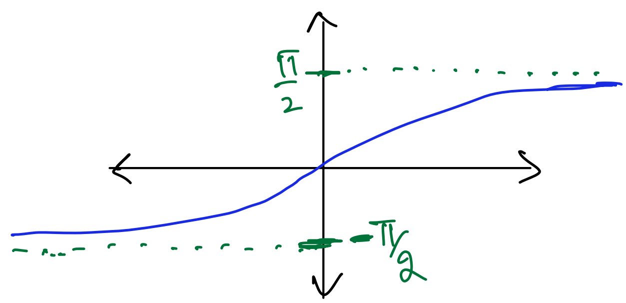
$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx + \lim_{s\to\infty} \int_0^s \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \left[\arctan(x) \Big|_t^0 \right] + \lim_{s\to\infty} \left[\arctan(x) \Big|_0^s \right]$$

$$= \lim_{t\to-\infty} -\arctan(t) + \lim_{s\to\infty} \arctan(s)$$

$$= -\left(-\frac{\pi}{2}\right) + \pi/2$$

$$= \pi$$



How to Compute Improper Integrals of Type II: Rewrite the integrals as follows:

- If f is only discontinuous at $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \left[\int_a^t f(x)\,dx \right].$$

- If f is only discontinuous at $x = a$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \left[\int_t^b f(x)\,dx \right].$$

- If f is only discontinuous at $x = c$, where $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 2: Determine whether the following integrals are convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

$\longrightarrow$ discontinuous at $x=2$

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$\int_t^5 \frac{1}{\sqrt{x-2}}\,dx \underset{\substack{u=x-2\\ du=dx}}{=} \int_{t-2}^3 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_{t-2}^3$$

$$= 2\sqrt{3} - 2\sqrt{t-2}$$

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} 2\sqrt{3} - 2\sqrt{t-2} = 2\sqrt{3} - 2\sqrt{2-2}$$

$$= 2\sqrt{3} \quad \text{(convergent)}$$

(b) $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$

$$= \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$$

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \ln|x-1|\Big|_0^t$$

$$= \lim_{t\to 1^-} \left(\ln|t-1| - \ln|0-1|\right) = \lim_{t\to 1^-} \ln|t-1| = -\infty$$

divergent

$$\int_1^3 \frac{1}{x-1}\,dx = \lim_{t\to 1^+} \int_t^3 \frac{1}{x-1}\,dx = \lim_{t\to 1^+} \ln|x-1|\Big|_t^3$$

$$\ln 2 - \lim_{t\to 1^+} \ln|t-1| = \infty$$

$$\Rightarrow \int_0^3 \frac{1}{x-1}\,dx \text{ is divergent}$$