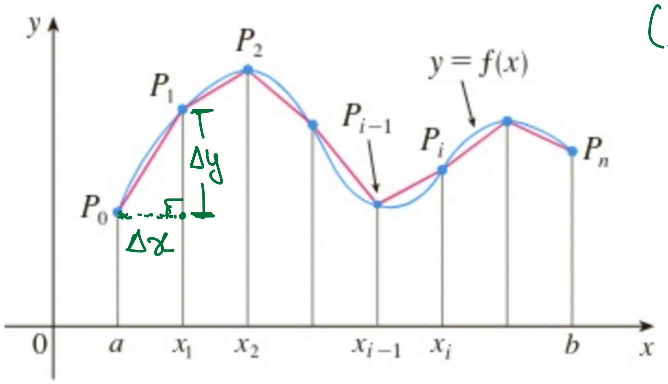
M16600 Lecture Notes
Section 8.1: Arc Length

■ **Section 8.1** textbook exercises, *page 589*: # 3, 5, 14, 11, 21.

How do we find the length of a curve $y = f(x)$, where $a \le x \le b$?



(length of curve)

$$L = \lim_{n\to\infty}\left(P_0P_1 + P_1P_2 + \cdots + P_{n-1}P_n\right)$$

By Pythagoras theorem,

$$P_{i-1}P_i = \sqrt{(\Delta y_i)^2 + (\Delta x)^2}$$

$$= \sqrt{\left(\frac{\Delta y_i}{\Delta x}\right)^2 + 1} \cdot \Delta x$$

when $n \to \infty$, $\Delta x \to dx$

$$\frac{\Delta y_i}{\Delta x} \to \frac{dy}{dx}$$

The Arc Length Formula. If $f'(x)$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$, where $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

or we can use Leibniz notation for derivatives and write the arc length formula as

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 1: Find the length of the curve $y = \frac{2}{3}x^{3/2}$ from the point $\underbrace{(1, \tfrac{2}{3})}_{(a,\, f(a))}$ to the point $\underbrace{(2, \tfrac{4}{3}\sqrt{2})}_{(b,\, f(b))}$.

$a = 1$, $b = 2$

$$y = \frac{2}{3}x^{3/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{3}\cdot\frac{3}{2}x^{1/2} = \sqrt{x}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + (\sqrt{x})^2}\, dx$$

$$= \int_1^2 \sqrt{1 + x}\, dx$$

$u = 1+x$
$du = dx$

$$\Rightarrow L = \int_2^3 \sqrt{u}\, du = \left.\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_2^3$$

$$= \frac{2}{3} u^{3/2}\Big|_2^3 = \frac{2}{3} u\sqrt{u}\Big|_2^3$$

$$= \frac{2}{3}\left(3\sqrt{3} - 2\sqrt{2}\right)$$

Example 2: Find the exact length of the curve $y = \ln(\sec x)$, where $0 \le x \le \pi/4$.

$$L = \int_0^{\pi/4} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \,\mathrm{Tan}\, x$$

$$= \mathrm{Tan}\, x$$

$$= \int_0^{\pi/4} \sqrt{1+\mathrm{Tan}^2 x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx$$

$$= \ln\left|\sec x + \mathrm{Tan}\, x\right|\Big|_0^{\pi/4}$$

$$= \ln\left|\sec\tfrac{\pi}{4} + \mathrm{Tan}\tfrac{\pi}{4}\right| - \ln\left|\sec 0 + \mathrm{Tan}\, 0\right|$$

$$= \ln\left|\sqrt{2}+1\right| - \ln\left|1+0\right|$$

$$= \ln\left(\sqrt{2}+1\right) - \ln 1$$

$$= \ln\left(\sqrt{2}+1\right)$$