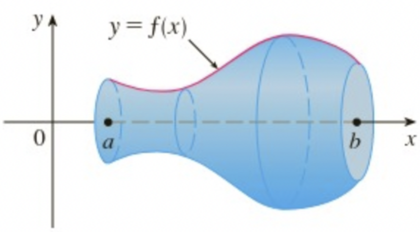
M16600 Lecture Notes

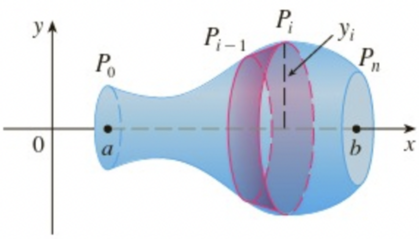
Section 8.2: Area of a Surface of Revolution

■ **Section 8.2** textbook exercises, *page 595*: # 1, 2, 3, 7.

A ***surface of revolution*** is formed when a curve is rotated about a line. How do we find the area of such a surface?



(a) Surface of revolution



(b) Approximating band

The area of the i band is $2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\Delta x$. See the discussion on page 591–592 of the textbook for more detail. Then an approximation of the surface area is

$$\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\Delta x$$

Thus, the surface area is

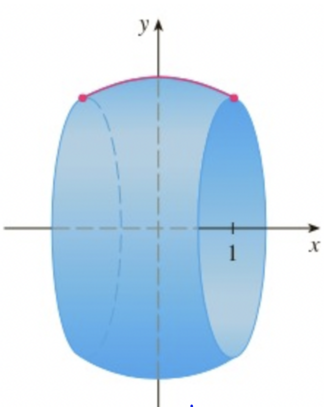
$$\lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\Delta x$$

$$= \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx.$$

Area of a Surface of Revolution about the x-axis. The surface area of a surface obtained by rotating the curve $y = y(x)$, $a \le x \le b$, about the x-axis is

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example 1: The curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2+y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis.



$$y = \sqrt{4-x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}}\cdot(-2x)$$

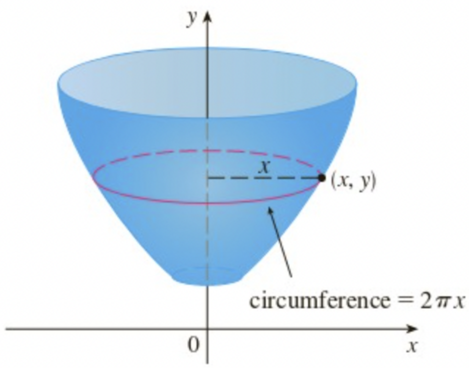
$$= \frac{-x}{\sqrt{4-x^2}}$$

$$S = \int_{-1}^{1} 2\pi\sqrt{4-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\,dx$$

$$= \int_{-1}^{1} 2\pi \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx = 2\pi\int_{-1}^{1}\sqrt{4-x^2}\sqrt{\frac{4-\cancel{x^2}+\cancel{x^2}}{4-x^2}}\,dx$$

$$= 2\pi\int_{-1}^{1}\sqrt{4-x^2}\sqrt{\frac{4}{4-x^2}}\,dx = 2\pi\int_{-1}^{1}\cancel{\sqrt{4-x^2}}\,\frac{2}{\cancel{\sqrt{4-x^2}}}\,dx$$

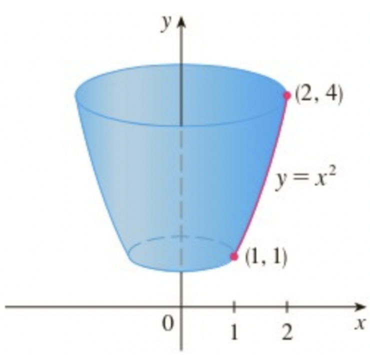
$$= 2\pi\int_{-1}^{1} 2\,dx = 2\pi\cdot 2\cdot(1-(-1)) = 8\pi$$



Area of a Surface of Revolution about the y-axis.
The surface area of a surface obtained by rotating the curve $y = y(x)$, $a \le x \le b$, about the y-axis is

$$S = \int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example 2: The arc of the parabola $y = x^2$ from $\underset{\substack{\uparrow\\ a}}{(1,1)}$ to $\underset{\substack{\uparrow\\ b}}{(2,4)}$ is rotated about the y-axis. Find the area of the resulting surface.



$$S = \int_1^2 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$$

$$S = \int_1^2 2\pi x\sqrt{1+(2x)^2}\,dx = 2\pi\int_1^2 x\sqrt{1+4x^2}\,dx$$

$$u = 1+4x^2 \Rightarrow du = 8x\,dx \Rightarrow \frac{1}{8}du = x\,dx$$

$$S = 2\pi\int_{1+4(1)^2}^{1+4(2)^2}\sqrt{u}\left(\frac{1}{8}du\right) = \frac{2\pi}{8}\int_5^{17}\sqrt{u}\,du$$

$$= \frac{2\pi}{8} \left. \frac{u^{3/2}}{3/2} \right|_5^{17} = \frac{2\pi}{8} \cdot \frac{2}{3} \cdot \left[17^{3/2} - 5^{3/2} \right]$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5} \right)$$